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TRIGONOMETRY FOR BEGINNERS.

SCHOOL CLASS BOOKS
CAMBRIDGE.

TRIGONOMETRY

FOR BEGINNERS

WITH NUMEROUS EXAMPLES.

BY

I. TODHUNTER, M.A., F.R.S.

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PREFACE.

THE present work is intended to be placed in the hands of beginners, and to serve as an introduction to the larger treatise on Plane Trigonometry, published by the author. The same plan has been adopted as in the *Algebra for Beginners:* the subject is discussed in short chapters, and a collection of examples is attached to each chapter. Many of these examples are original and have been constructed with reference to the most important points and to the usual difficulties of beginners; the rest have been derived from College and University Examination Papers.

Especial attention has been paid to the numerical calculations which occur in Trigonometry, in order that the work may be suitable for those who wish to confine themselves to the practical solution of triangles, as well as for those who intend to advance in the study of mathematics.

The subject is arranged in the order which appears most convenient for beginners; an acquaintance with the books of Euclid which are usually read, and with the rudiments of Algebra, being all that is assumed. The first fourteen chapters present the geometrical part of Plane Trigonometry; they contain all that is necessary for practical purposes. The remaining chapters are of a more analytical character, and are important in the Theory of Mathematics. It will be found that the order of study may be varied at the discretion of the teacher, and the theoretical part taken at an earlier period.

The range of matter included is such as seems required by the various examinations in Elementary Trigonometry which are now carried on in the country; it is hoped that nothing has been omitted which usually finds a place in such examinations.

Any remarks relating to the book, and especially the indication of difficulties or omissions in the text or the examples, will be most thankfully received.

I. TODHUNTER.

CAMBRIDGE,
Oct. 1866.

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TRIGONOMETRY FOR BEGINNERS.

I. *Measurement of Angles by Degrees or Grades.*

1. THE word Trigonometry is derived from two Greek words, one signifying a *triangle*, and the other signifying *I measure*. Plane Trigonometry originally denoted the science in which the relations subsisting between the sides and the angles of a plane triangle were investigated, and the modes of investigation were almost entirely geometrical. But now the term Plane Trigonometry has a wider meaning, and comprises investigations with respect to plane angles whether forming a triangle or not, and the investigations are made by the aid of algebraical symbols and formulæ. Before beginning the present treatise the student should therefore become acquainted with Algebra, at least as far as the solution of simple equations. The parts of the elements of Euclid which are usually read are also necessary.

2. We have first to explain how angles are measured. Some angle is selected as the *unit*, and the measure of any other angle is the number of units which it contains. Any angle might be taken for the *unit*, as for example a *right angle;* but a smaller angle than a right angle is found more convenient. Accordingly a right angle is divided into 90 equal parts called degrees; and any angle may be estimated by ascertaining the number of degrees which it contains. If the angle does not contain an exact number of degrees we can express it in degrees and a fraction of a degree. A degree is divided into 60 equal parts called minutes, and a minute into 60 equal parts called seconds; and thus a

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fraction of a degree may if we please be converted into minutes and seconds.

3. Thus, for example, half a right angle contains 45 degrees; a quarter of a right angle contains 22½ degrees, which we may write in the decimal notation 22·5 degrees, or we may express it as 22 degrees 30 minutes; one-sixteenth of a right angle contains 5⅝ degrees, that is, 5·625 degrees, or 5 degrees, 37 minutes, 30 seconds.

4. Symbols are used as abbreviations of the words *degrees*, *minutes*, and *seconds*. Thus $5^\circ\ 37'\ 30''$ is used to denote 5 degrees, 37 minutes, 30 seconds.

5. The method of estimating angles by degrees, minutes, and seconds, is almost universally adopted in practical calculations. Another method was proposed in France, towards the end of the last century, in connexion with a uniform system of decimal tables of weights and measures. In this method a right angle is divided into 100 equal parts called *grades*, a grade is divided into 100 equal parts called *minutes*, and a minute is divided into 100 equal parts called *seconds*. On account of the occurrence of the number *one hundred* in forming the subdivisions of a right angle, this method of estimating angles is called the *centesimal* method; and the common method is called the *sexagesimal* method, on account of the occurrence of the number *sixty* in forming the subdivisions of a degree. The centesimal method is also sometimes called the *French* method, and the sexagesimal method is called the *English* method.

6. Symbols are used as abbreviations of the words *grades*, *minutes*, and *seconds*, in the centesimal method. Thus, $5^g\ 37^{\backprime}\ 30^{\backprime\backprime}$ is used to denote 5 grades, 37 minutes, 30 seconds in the centesimal method. A centesimal minute and second are not the same as a sexagesimal minute and second, and the accents or dashes which are used to denote minutes and seconds in the two methods are distinguished by sloping in different directions.

7. In the centesimal method any whole number of minutes and seconds may be expressed immediately as a decimal fraction of a grade. Thus, 37 centesimal minutes

are $\frac{37}{100}$ of a grade, that is, ·37 of a grade; and 30 centesimal seconds are $\frac{30}{(100)^2}$ of a grade, that is, ·003 of a grade. Hence $5^g\ 37'\ 30''$ may be written as $5^g{\cdot}373$; and since a grade is $\left(\frac{1}{100}\right)^{\text{th}}$ of a right angle, $5^g{\cdot}373$ may be written as ·05373 of a right angle. Notwithstanding this great advantage of the centesimal method, the sexagesimal method has been retained in practical calculations, because the latter had become thoroughly established by long use in mathematical works, and especially in mathematical tables, before the former was proposed; and such works and tables would have been rendered almost useless by the change in the method of estimating angles. The centesimal method is not practically used even in France.

8. Although the centesimal method is not used in practical calculations it is customary to give an account of the method in works on Trigonometry; and it is shewn how to compare the numbers which measure the same angle in the English and French methods. This we shall explain in the next three Articles.

9. *To compare the number of degrees in any angle with the number of grades in the same angle.*

Let D be the number of *degrees* in any given angle, G the number of *grades* in the same angle. Then, since there are 90 degrees in a right angle, $\frac{D}{90}$ expresses the ratio of the given angle to a right angle; and, since there are 100 grades in a right angle, $\frac{G}{100}$ also expresses the ratio of the given angle to a right angle.

Hence $$\frac{D}{90}=\frac{G}{100};$$

therefore $$D=\frac{90}{100}G=\frac{9}{10}G=G-\frac{1}{10}G,$$

and $$G=\frac{100}{90}D=\frac{10}{9}D=D+\frac{1}{9}D.$$

The formula $D=G-\frac{1}{10}G$ gives the following rule: *From the number of grades in any angle subtract one-tenth of that number; the remainder is the number of degrees in the angle.*

The formula $G=D+\frac{1}{9}D$ gives the following rule: *To the number of degrees in any angle add one-ninth of that number; the sum is the number of grades in the angle.*

10. *To compare the number of English minutes in any angle with the number of French minutes in the same angle.*

Let m be the number of English minutes in any angle, μ the number of French minutes in the same angle. Then, since there are 90×60 English minutes in a right angle, $\frac{m}{90\times 60}$ expresses the ratio of the given angle to a right angle; and since there are 100×100 French minutes in a right angle $\frac{\mu}{100\times 100}$ also expresses the ratio of the given angle to a right angle.

Hence
$$\frac{m}{90\times 60}=\frac{\mu}{100\times 100};$$
therefore
$$m=\frac{9\times 6}{10\times 10}\mu=\frac{27}{50}\mu,$$
and
$$\mu=\frac{50}{27}m.$$

11. Similarly, if s be the number of English seconds in any angle, and σ the number of French seconds in the same angle
$$\frac{s}{90\times 60\times 60}=\frac{\sigma}{100\times 100\times 100};$$
therefore
$$s=\frac{9\times 6\times 6}{10\times 10\times 10}\sigma=\frac{81}{250}\sigma,$$
$$\sigma=\frac{250}{81}s.$$

EXAMPLES. I.

Express the following six angles in the French mode:

1. 54^{0}. 2. $1^{0}\ 21'$.
3. $6^{0}\ 18'$. 4. $9^{0}\ 49'\ 57''$.
5. $27^{0}\ 41'\ 51''$. 6. $67^{0}\cdot4325$.

Express the following six angles in the English mode:

7. 30^{g}. 8. $3^{g}\ 50^{\backprime}$.
9. $10^{g}\ 42^{\backprime}\ 50^{\backprime\backprime}$. 10. $20^{g}\ 77^{\backprime}\ 50^{\backprime\backprime}$.
11. $31^{g}\ 7^{\backprime}\ 50^{\backprime\backprime}$. 12. $76^{g}\cdot452$.

Express the following six angles in both modes:

13. $\frac{1}{5}$ of a right angle. 14. $\frac{5}{8}$ of a right angle.

15. $\frac{11}{16}$ of a right angle. 16. $\frac{45}{64}$ of a right angle.

17. The angle of an equilateral triangle.

18. The angle at the vertex of the isosceles triangle described in Euclid IV. 10.

19. The sum of two angles is 30 grades, and their difference is 9 degrees: find each angle.

20. The difference of the two acute angles of a right-angled triangle is 20 grades: find the angles in degrees.

21. Find the number of English minutes in a grade.

22. Find the number of English seconds in a French minute.

23. Find the number of French minutes in a degree.

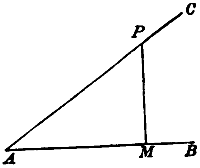
24. Find the number of French seconds in an English minute.

25. Find the ratio of an angle of $1^{0}\ 25'$ to an angle of $1^{g}\ 25^{\backprime}$.

II. *Trigonometrical Ratios.*

12. There are certain quantities connected with an angle which are called the *Trigonometrical Ratios* of the angle; in the present Chapter we shall define these Trigonometrical Ratios, and demonstrate some of their most important properties, confining ourselves to angles less than a right angle. It will be seen as we proceed with the book that the whole subject rests on the definitions and properties contained in the present Chapter.

13. Let BAC be any acute angle; take any point in either of the containing straight lines, and from it draw a perpendicular to the other straight line: let P be the point in AC, and PM perpendicular to AB. We shall use the letter A to denote the angle BAC.



The following are the definitions of the Trigonometrical Ratios of the angle A:

$\frac{PM}{AP}$, that is $\frac{\textit{perpendicular}}{\textit{hypotenuse}}$, is called the *sine* of A;

$\frac{AM}{AP}$, that is $\frac{\textit{base}}{\textit{hypotenuse}}$, is called the *cosine* of A;

$\frac{PM}{AM}$, that is $\frac{\textit{perpendicular}}{\textit{base}}$, is called the *tangent* of A;

$\frac{AM}{PM}$, that is $\frac{\textit{base}}{\textit{perpendicular}}$, is called the *cotangent* of A;

$\frac{AP}{AM}$, that is $\frac{\textit{hypotenuse}}{\textit{base}}$, is called the *secant* of A;

$\frac{AP}{PM}$, that is $\frac{\textit{hypotenuse}}{\textit{perpendicular}}$, is called the *cosecant* of A.

When the cosine of A is subtracted from unity the remainder is called the *versed sine* of A. When the sine of A is subtracted from unity the remainder is called the *coversed* sine of A. But the term *versed sine* is not often used, and the term *coversed sine* is scarcely ever used.

14. The words *sine*, *cosine*, &c. are usually abbreviated in writing and printing; thus the above definitions may be expressed as follows:

$$\sin A = \frac{PM}{AP}, \qquad \cos A = \frac{AM}{AP},$$

$$\tan A = \frac{PM}{AM}, \qquad \cot A = \frac{AM}{PM},$$

$$\sec A = \frac{AP}{AM}, \qquad \operatorname{cosec} A = \frac{AP}{PM},$$

$$\operatorname{vers} A = 1 - \cos A, \qquad \operatorname{covers} A = 1 - \sin A.$$

15. The *sine*, *cosine*, *tangent*, *cotangent*, *secant*, *cosecant*, *versed sine*, and *coversed sine* of an angle are called the *Trigonometrical Ratios* of the angle: it will be seen from the definitions that the term *ratio* is appropriate, because each of the quantities defined is the *ratio* of one length to another, that is, each of the quantities is some arithmetical number or fraction. The *Trigonometrical Ratios* have been sometimes called *Trigonometrical Functions*, and sometimes *Goniometrical Ratios* or *Functions*.

16. The excess of a right angle over any angle is called the *complement* of that angle. Thus if A be the number of degrees in any angle, $90 - A$ is the number of degrees in the complement of the angle. This affords another method of defining some of the Trigonometrical Ratios: after defining, as in Art. 14, the *sine*, *tangent*, and *secant* of an angle we may say:

the cosine of an angle is the sine of the complement of that angle;

the cotangent of an angle is the tangent of the complement of that angle;

the cosecant of an angle is the secant of the complement of that angle.

For in the triangle PAM the angle APM is the complement of the angle A; and

$$\sin APM = \frac{\textit{perpendicular}}{\textit{hypotenuse}} = \frac{AM}{AP} = \cos A;$$

$$\tan APM = \frac{\textit{perpendicular}}{\textit{base}} = \frac{AM}{MP} = \cot A;$$

$$\sec APM = \frac{\textit{hypotenuse}}{\textit{base}} = \frac{AP}{MP} = \operatorname{cosec} A.$$

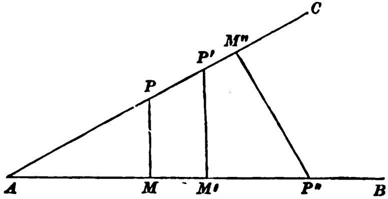
These results may also be expressed thus:

the sine of an angle is the cosine of the complement of that angle;

the tangent of an angle is the cotangent of the complement of that angle;

the secant of an angle is the cosecant of the complement of that angle.

17. *The Trigonometrical Ratios remain unchanged so long as the angle remains unchanged.*



Let BAC be any angle; in AC take any point P, and draw PM perpendicular to AB; also take any other point P', and draw $P'M'$ perpendicular to AB. Then, by similar triangles, Euclid VI. 4,

$$\frac{PM}{AP} = \frac{P'M'}{AP'};$$

that is, the *sine* of the angle A is the same whether it be formed from the triangle APM or from the triangle $AP'M'$.

The same result holds for the other Trigonometrical Ratios.

Or we may suppose a point P'' taken in AB and $P''M''$ drawn perpendicular to AC; then the triangles APM and $AP''M''$ are similar, and

$$\frac{PM}{AP} = \frac{P''M''}{AP''}.$$

18. We have now defined the Trigonometrical Ratios, and have shewn that each Ratio has only one value so long as the angle is unchanged: we proceed to establish certain relations which hold among the Trigonometrical Ratios.

19. We have immediately from the definitions

$$\tan A \times \cot A = \frac{PM}{AM} \times \frac{AM}{PM} = 1,$$

therefore $$\tan A = \frac{1}{\cot A}, \quad \cot A = \frac{1}{\tan A};$$

$$\sec A \times \cos A = \frac{AP}{AM} \times \frac{AM}{AP} = 1,$$

therefore $$\sec A = \frac{1}{\cos A}, \quad \cos A = \frac{1}{\sec A};$$

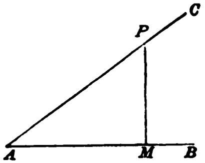
$$\operatorname{cosec} A \times \sin A = \frac{AP}{PM} \times \frac{PM}{AP} = 1,$$

therefore $$\operatorname{cosec} A = \frac{1}{\sin A}, \quad \sin A = \frac{1}{\operatorname{cosec} A}.$$

Also $$\tan A = \frac{PM}{AM} = \frac{PM}{AP} \div \frac{AM}{AP} = \frac{\sin A}{\cos A};$$

$$\cot A = \frac{AM}{PM} = \frac{AM}{AP} \div \frac{PM}{AP} = \frac{\cos A}{\sin A}.$$

20. *To shew that* $(\sin A)^2 + (\cos A)^2 = 1$.



In the right-angled triangle APM we have

$$PM^2 + AM^2 = AP^2;$$

therefore $$\frac{PM^2 + AM^2}{AP^2} = 1,$$

therefore $$\frac{PM^2}{AP^2} + \frac{AM^2}{AP^2} = 1,$$

therefore $$\left(\frac{PM}{AP}\right)^2 + \left(\frac{AM}{AP}\right)^2 = 1;$$

that is, $$(\sin A)^2 + (\cos A)^2 = 1.$$

21. With respect to the preceding demonstration it should be remarked that it is shewn in Euclid I. 47, that the square described on the hypotenuse of a right-angled triangle is equal to the sum of the squares described on the sides; and it is known that the *geometrical* square described on any straight line is measured by the *arithmetical* square of the number which measures the length of the straight line. From combining these two results we obtain the arithmetical equality $PM^2 + AM^2 = AP^2$, which is the foundation of the preceding demonstration.

22. It is usual for shortness to write $(\sin A)^2$ thus, $\sin^2 A$; similarly $(\sin A)^3$ is written thus, $\sin^3 A$. The same mode of abbreviation is used for the powers of the other Trigonometrical Ratios; and so the result obtained in Art. 20 is usually written thus,

$$\sin^2 A + \cos^2 A = 1.$$

23. *To shew that*

$(\sec A)^2 = 1 + (\tan A)^2$, *and* $(\operatorname{cosec} A)^2 = 1 + (\cot A)^2$.

In the right-angled triangle APM (see fig. to Art. 20)

we have $$AP^2 = AM^2 + PM^2,$$

therefore $$\frac{AP^2}{AM^2} = 1 + \frac{PM^2}{AM^2},$$

therefore $$\left(\frac{AP}{AM}\right)^2 = 1 + \left(\frac{PM}{AM}\right)^2;$$

that is, $$(\sec A)^2 = 1 + (\tan A)^2.$$

Again $$AP^2 = PM^2 + AM^2;$$

therefore $$\left(\frac{AP}{PM}\right)^2 = 1 + \left(\frac{AM}{PM}\right)^2;$$

that is, $$(\operatorname{cosec} A)^2 = 1 + (\cot A)^2.$$

The results here obtained are usually written thus,

$$\sec^2 A = 1 + \tan^2 A, \qquad \operatorname{cosec}^2 A = 1 + \cot^2 A.$$

24. By means of the relations which have been established in Arts. 19—23 we can express all the Trigonometrical Ratios in terms of any one of them.

Thus, for example, we will express all the rest in terms of the *sine:*

$$\cos A = \sqrt{(1 - \sin^2 A)} \qquad \text{(Art. 20)},$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{\sin A}{\sqrt{(1 - \sin^2 A)}} \qquad \text{(Arts. 19, 20)},$$

$$\cot A = \frac{\cos A}{\sin A} = \frac{\sqrt{(1 - \sin^2 A)}}{\sin A} \qquad \text{(Arts. 19, 20)},$$

$$\sec A = \frac{1}{\cos A} = \frac{1}{\sqrt{(1 - \sin^2 A)}} \qquad \text{(Arts. 19, 20)},$$

$$\operatorname{cosec} A = \frac{1}{\sin A} \qquad \text{(Art. 19)},$$

$$\text{vers}\,A = 1 - \cos A = 1 - \sqrt{(1 - \sin^2 A)} \quad \text{(Art. 20).}$$

Again, we will express all the rest in terms of the *tangent:*

$$\sin A = \frac{1}{\text{cosec}\,A} = \frac{1}{\sqrt{(1 + \cot^2 A)}} = \frac{1}{\sqrt{\left(1 + \frac{1}{\tan^2 A}\right)}}$$

$$= \frac{\tan A}{\sqrt{(1 + \tan^2 A)}} \quad \text{(Arts. 19, 23),}$$

$$\cos A = \frac{1}{\sec A} = \frac{1}{\sqrt{(1 + \tan^2 A)}} \quad \text{(Arts. 19, 23),}$$

$$\cot A = \frac{1}{\tan A} \quad \text{(Art. 19),}$$

$$\sec A = \sqrt{(1 + \tan^2 A)} \quad \text{(Art. 23),}$$

$$\text{cosec}\,A = \frac{1}{\sin A} = \frac{\sqrt{(1 + \tan^2 A)}}{\tan A},$$

$$\text{vers}\,A = 1 - \cos A = 1 - \frac{1}{\sqrt{(1 + \tan^2 A)}}.$$

25. If we have given the value of one of the Trigonometrical Ratios we can thus find the values of the rest.

Suppose, for example, that $\sin A = \frac{3}{5}$; then we have

$$\cos A = \sqrt{(1 - \sin^2 A)} = \sqrt{\left(1 - \frac{9}{25}\right)} = \sqrt{\frac{16}{25}} = \frac{4}{5},$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{3}{5} \div \frac{4}{5} = \frac{3}{5} \times \frac{5}{4} = \frac{3}{4}, \quad \cot A = \frac{1}{\tan A} = \frac{4}{3},$$

$$\sec A = \frac{1}{\cos A} = \frac{5}{4}, \quad \text{cosec}\,A = \frac{1}{\sin A} = \frac{5}{3},$$

$$\text{vers}\,A = 1 - \cos A = 1 - \frac{4}{5} = \frac{1}{5}.$$

EXAMPLES. II.

Find the values of the other Trigonometrical Ratios in the following eight examples, having given:

1. $\sin A = \frac{12}{13}$. 2. $\sin A = \frac{40}{41}$. 3. $\cos A = {\cdot}28$.

4. $\cos A = \frac{60}{61}$. 5. $\tan A = \frac{4}{3}$. 6. $\sin A = \frac{1}{3}$.

7. $\sin A = \frac{2m}{m^2+1}$. 8. $\cos A = \frac{2mn}{m^2+n^2}$.

Demonstrate the following identities:

9. $(\sin A + \cos A)^2 + (\sin A - \cos A)^2 = 2$.

10. $\sin^2 A - \cos^2 B = \sin^2 B - \cos^2 A$.

11. $\sec^2 A \operatorname{cosec}^2 A = \sec^2 A + \operatorname{cosec}^2 A$.

12. $\sin^4 A + \cos^4 A = 1 - 2\sin^2 A \cos^2 A$.

13. $\tan A + \cot A = \sec A \operatorname{cosec} A$.

14. $\sin^4 A - \cos^4 A = \sin^2 A - \cos^2 A$.

15. $\sin^2 A \tan A + \cos^2 A \cot A = \frac{1 - 2\sin^2 A \cos^2 A}{\sin A \cos A}$.

16. $\sin^2 A + \operatorname{vers}^2 A = 2(1 - \cos A)$.

17. $\sin^3 A + \cos^3 A = (\sin A + \cos A)(1 - \sin A \cos A)$.

18. $\sin^6 A + \cos^6 A = \sin^4 A + \cos^4 A - \sin^2 A \cos^2 A$.

19. $\sin^2 A \tan^2 A + \cos^2 A \cot^2 A = \tan^2 A + \cot^2 A - 1$.

20. $\sin A \tan^2 A + \operatorname{cosec} A \sec^2 A - 2\tan A \sec A$
$= \operatorname{cosec} A - \sin A$.

21. $(\sin A \cos B + \cos A \sin B)^2$
$+ (\cos A \cos B - \sin A \sin B)^2 = 1$.

22. $(1 + \sin A + \cos A)^2 = 2(1 + \sin A)(1 + \cos A)$.

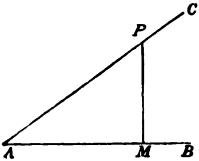
23. $(1 - \sin A - \cos A)^2 (1 + \sin A + \cos A)^2$
$= 4\sin^2 A \cos^2 A$.

24. $(1 + \sin A - \cos A)^2 + (1 + \cos A - \sin A)^2$
$= 4(1 - \sin A \cos A)$.

III. *Values of the Trigonometrical Ratios for an angle of* 45°, *of* 60°, *of* 30°.

26. In this Chapter we shall find the values of the Trigonometrical Ratios for certain angles.

27. *To determine the values of the Trigonometrical Ratios for an angle of* 45°.



Let BAC be an angle of 45°; take any point P in AC, and draw PM perpendicular to AB. Since PAM is half a right angle, APM is also half a right angle; therefore $PM = AM$.

Now $$PM^2 + AM^2 = AP^2;$$

thus $$2PM^2 = AP^2;$$

therefore $\left(\frac{PM}{AP}\right)^2 = \frac{1}{2}$; therefore $\frac{PM}{AP} = \frac{1}{\sqrt{2}}$.

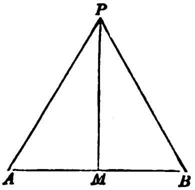
Thus $$\sin 45^\circ = \frac{PM}{AP} = \frac{1}{\sqrt{2}}; \quad \cos 45^\circ = \frac{AM}{AP} = \frac{1}{\sqrt{2}}.$$

$$\tan 45^\circ = \frac{PM}{AM} = 1; \quad \cot 45^\circ = \frac{AM}{PM} = 1.$$

$$\sec 45^\circ = \frac{AP}{AM} = \sqrt{2}; \quad \operatorname{cosec} 45^\circ = \frac{AP}{PM} = \sqrt{2}.$$

$$\operatorname{vers} 45^\circ = 1 - \cos 45^\circ = 1 - \frac{1}{\sqrt{2}}.$$

28. *To determine the values of the Trigonometrical Ratios for an angle of* 60^0 *and for an angle of* 30^0.



Let APB be an equilateral triangle, so that the angle PAB contains 60 degrees; draw PM perpendicular to AB, then $AM=MB$;

$$\text{therefore } AM=\tfrac{1}{2}AB=\tfrac{1}{2}AP.$$

$$\text{Thus}\quad \cos 60^0=\frac{AM}{AP}=\frac{1}{2};$$

$$\sin 60^0=\surd(1-\cos^2 60)=\surd\left(1-\frac{1}{4}\right)=\surd\frac{3}{4}=\frac{\surd 3}{2};$$

$$\tan 60^0=\frac{\sin 60^0}{\cos 60^0}=\frac{\surd 3}{2}\div\frac{1}{2}=\surd 3;\quad \cot 60^0=\frac{1}{\tan 60^0}=\frac{1}{\surd 3};$$

$$\sec 60^0=\frac{1}{\cos 60^0}=2;\qquad \operatorname{cosec} 60^0=\frac{1}{\sin 60^0}=\frac{2}{\surd 3};$$

$$\operatorname{vers} 60^0=1-\cos 60^0=1-\frac{1}{2}=\frac{1}{2}.$$

The Trigonometrical Ratios for an angle of 30^0 may be found by Art. 16: thus

$$\sin 30^0=\cos 60^0=\frac{1}{2};\qquad \cos 30^0=\sin 60^0=\frac{\surd 3}{2};$$

$$\tan 30^0=\cot 60^0=\frac{1}{\surd 3};\qquad \cot 30^0=\tan 60^0=\surd 3;$$

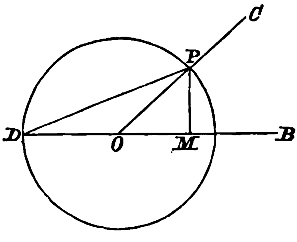
$$\sec 30^0=\operatorname{cosec} 60^0=\frac{2}{\surd 3};\quad \operatorname{cosec} 30^0=\sec 60^0=2;$$

$$\operatorname{vers} 30^0=1-\cos 30^0=1-\frac{\surd 3}{2}.$$

29. The Trigonometrical Ratios for any angle can be found approximately; the ratios are seldom capable of being expressed *exactly*, as they are in the special cases which we have here considered, but the calculations may be carried to any assigned degree of accuracy. We shall not enter into an account of the processes of calculation in the present work, but may refer to the more complete treatise. It will be sufficient to state as a fact that tables may be easily procured which give to seven places of decimals the sine of any angle which can be expressed in degrees and minutes; the other ratios can be determined when the sine is known.

30. Although we shall not explain the mode in which the tables are constructed, yet the student will readily see as he proceeds with the subject that various formulæ occur which might be useful in calculating the values of the Trigonometrical Ratios. Especially he may notice the formulæ hereafter to be given by which we may determine the Trigonometrical Ratios for an angle which is the sum or the difference of two other angles having known Trigonometrical Ratios. And we shall give a formula in the next Article which will enable us to determine the Trigonometrical Ratios for the *half* of an angle when the Trigonometrical Ratios of the angle itself are known.

31. *To express the tangent of half an angle in terms of the sine and cosine of the angle.*



Let *BOC* be any angle, which we will denote by *A*.

Take any point P in OC. With O as centre, and OP as radius, describe a circle. Produce BO to meet the circumference at D. Draw PM perpendicular to OB; and join PD.

By Euclid III. 20 the angle $PDM = \frac{1}{2}A$.

Now $\tan PDM = \dfrac{PM}{DM} = \dfrac{PM}{DO+OM} = \dfrac{PM}{OP+OM}$.

Let $OP = a$; then

$$\frac{PM}{OP} = \sin A, \text{ therefore } PM = a \sin A,$$

$$\frac{OM}{OP} = \cos A, \text{ therefore } OM = a \cos A.$$

Thus $\tan PDM = \dfrac{a \sin A}{a + a \cos A} = \dfrac{\sin A}{1 + \cos A}$;

that is, $$\tan \tfrac{1}{2}A = \frac{\sin A}{1 + \cos A}.$$

32. By the preceding Article when $\sin A$ and $\cos A$ are known we can determine $\tan \frac{1}{2}A$; and then by Art. 24 we can deduce from $\tan \frac{1}{2}A$ the values of the other trigonometrical ratios of $\frac{1}{2}A$.

For example, suppose $A = 30^0$, then $\frac{1}{2}A = 15^0$.

$$\tan 15^0 = \frac{\sin 30^0}{1 + \cos 30^0} = \frac{\frac{1}{2}}{1 + \frac{\sqrt{3}}{2}} = \frac{1}{2 + \sqrt{3}};$$

we may multiply both numerator and denominator of the last fraction by $2 - \sqrt{3}$, and thus we obtain the more convenient result $\tan 15^0 = 2 - \sqrt{3}$.

By Art. 23 $\sec^2 15^0 = 1 + \tan^2 15 = 1 + (2 - \sqrt{3})^2 = 8 - 4\sqrt{3}$.

Hence we find $\sec 15^0$ by taking the square root of $8 - 4\sqrt{3}$; it is shewn in Algebra how to extract this square root; it is easy, by squaring both members, to verify that

$$\sqrt{8 - 4\sqrt{3}} = (\sqrt{3} - 1)\sqrt{2}.$$

Hence $$\sec 15^0 = (\surd 3 - 1)\surd 2.$$

$$\cos 15^0 = \frac{1}{(\surd 3 - 1)\surd 2} = \frac{\surd 3 + 1}{(\surd 3 + 1)(\surd 3 - 1)\surd 2} = \frac{\surd 3 + 1}{2\surd 2}.$$

$$\cot 15^0 = \frac{1}{\tan 15^0} = 2 + \surd 3.$$

$$\operatorname{cosec}^2 15^0 = 1 + \cot^2 15 = 8 + 4\surd 3,$$

$$\operatorname{cosec} 15^0 = (\surd 3 + 1)\surd 2.$$

$$\sin 15^0 = \frac{1}{(\surd 3 + 1)\surd 2} = \frac{\surd 3 - 1}{2\surd 2}.$$

Since the Trigonometrical Ratios for an angle of 15⁰ are known we can immediately deduce those for an angle of 75⁰ by Art. 16.

33. The student should render himself perfectly familiar with the values of the Trigonometrical Ratios for an angle of 30⁰, 45⁰, or 60⁰; as they will be perpetually used in the subject. Thus, for example, if an angle of 60⁰ occurs it may be necessary to have the cosine of this angle, which has been found to be $\frac{1}{2}$. And conversely, if the cosine of an angle is known to be $\frac{1}{2}$, and the angle is less than a right angle, the student will immediately infer that the angle contains 60⁰. Should there be any difficulty in this inference it will be removed by the remarks made hereafter, in which it will appear why we introduce the restriction that *the angle is less than a right angle.*

34. It may be observed that if an angle be less than 45⁰ the cosine of the angle is greater than the sine, and if the angle be greater than 45⁰ and less than 90⁰ the cosine is less than the sine: these results follow immediately from the figure in Art. 13, since the greater side in a triangle is opposite to the greater angle.

35. From the result given in Art. 31 we can deduce some other results which will be useful hereafter, while the process will serve to apply some of the formulæ already established.

We have $$(\tan \tfrac{1}{2}A)^2 = \frac{\sin^2 A}{(1 + \cos A)^2};$$

hence, by Art. 20,

$$(\tan \tfrac{1}{2}A)^2 = \frac{1-\cos^2 A}{(1+\cos A)^2} = \frac{(1-\cos A)(1+\cos A)}{(1+\cos A)^2}.$$

Thus $$(\tan \tfrac{1}{2}A)^2 = \frac{1-\cos A}{1+\cos A} \text{ .... } (1).$$

But, by Art. 23,

$$(\sec \tfrac{1}{2}A)^2 = 1 + (\tan \tfrac{1}{2}A)^2$$
$$= 1 + \frac{1-\cos A}{1+\cos A} = \frac{2}{1+\cos A};$$

therefore, by Art. 19,

$$(\cos \tfrac{1}{2}A)^2 = \frac{1+\cos A}{2} \text{ ....} (2),$$

and $$(\sin \tfrac{1}{2}A)^2 = 1 - (\cos \tfrac{1}{2}A)^2 = 1 - \frac{1+\cos A}{2}$$
$$= \frac{1-\cos A}{2} \text{ ....} (3).$$

And $$\sin A = (1+\cos A)\tan \tfrac{1}{2}A$$
$$= 2(\cos \tfrac{1}{2}A)^2 \frac{\sin \tfrac{1}{2}A}{\cos \tfrac{1}{2}A} = 2\sin \tfrac{1}{2}A \cos \tfrac{1}{2}A \text{ ...} (4).$$

In these Articles we use the form $(\tan \frac{1}{2}A)^2$ as most intelligible for a beginner; but for abbreviation this is commonly written thus, $\tan^2 \frac{1}{2}A$; similarly, $\sin^2 \frac{1}{2}A$ is written for $(\sin \frac{1}{2}A)^2$.

The results contained in (2) and (3) may be presented thus:

$$\cos A = 2(\cos \tfrac{1}{2}A)^2 - 1 = 1 - 2(\sin \tfrac{1}{2}A)^2 = (\cos \tfrac{1}{2}A)^2 - (\sin \tfrac{1}{2}A)^2.$$

Suppose we put $2B$ for A in (4); we thus obtain

$$\sin 2B = 2\sin B \cos B \text{ ....} (5).$$

The student will have to accustom himself to such changes as we have here exemplified; he must regard (4) and (5) as expressing under slightly different forms exactly the same result, so that from either form the other immediately follows.

Another demonstration of the results obtained in Art. 31 and in the present Article will be found in Art. 184.

EXAMPLES. III.

Find A from the following equations:

1. $3 \sin A = 2 \cos^2 A$.
2. $\sec A \tan A = 2\sqrt{3}$.
3. $\sec^2 A - \frac{5}{2} \sec A + 1 = 0$.
4. $6 \cot^2 A - 4 \cos^2 A = 1$.
5. $3 \operatorname{cosec}^2 A + 8 \sin^2 A = 10$.
6. $\tan^2 A - 4 \tan A + 1 = 0$.

Find A and B from the following equations:

7. $\dfrac{\sin A}{\sin B} = \sqrt{2}, \quad \dfrac{\cos A}{\cos B} = \dfrac{\sqrt{2}}{\sqrt{3}}$.
8. $\dfrac{\sin A}{\cos B} = \dfrac{\sqrt{3}}{\sqrt{2}}, \quad \dfrac{\cos A}{\sin B} = \dfrac{1}{\sqrt{2}}$.
9. $\cos (A - B) = \dfrac{\sqrt{3}}{2}, \quad \sin (A - B) = \cos (A + B)$.
10. $\cos (2A + B) = \dfrac{1}{2}, \quad \sin (3A - B) = \dfrac{1}{2}$.
11. $\tan (4A + 7B) = 2 + \sqrt{3}, \quad \tan (5A - 7B) = 2 - \sqrt{3}$.
12. $\sin A + \sin B = \sqrt{2}, \quad \sin^2 A + \sin^2 B = 1$.
13. Find A, B, and C from the equations

$$\cos (A + B + C) = \frac{1}{2}, \quad \sin (A + B - C) = \frac{1}{2}, \quad \tan (B + C) = 1.$$

14. Find the Trigonometrical Ratios for an angle of $22\frac{1}{2}°$.

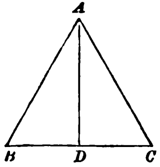
15. Find $\tan 7\frac{1}{2}°$. 16. Find $\tan 37\frac{1}{2}°$.

IV. *Applications of Trigonometry.*

36. In the present Chapter we shall give some examples of the use of the Trigonometrical Ratios. It will not be possible to supply any great variety or extent of illustration, because at present we have not advanced beyond the simplest elements of the subject; but the student may be led to take more interest in Trigonometry from seeing even at this early stage that it admits of valuable practical applications.

We begin by demonstrating an important proposition, which connects the sides of a triangle with the Trigonometrical Ratios of the angles.

37. *In a triangle the sides are proportional to the sines of the opposite angles.*



Let ABC be a triangle; from A draw AD perpendicular to the opposite side, meeting that side at D.

Thus $\frac{AD}{AB} = \sin B$, therefore $AD = AB \sin B$;

and $\frac{AD}{AC} = \sin C$, therefore $AD = AC \sin C$;

therefore $AB \sin B = AC \sin C$;

therefore $$\frac{AB}{AC} = \frac{\sin C}{\sin B}.$$

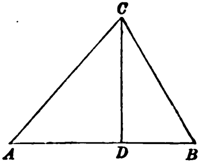
This shews that the proposition is true for any pair of sides.

We suppose that the two angles considered are *acute;* this will be sufficient for the applications we shall make in the present Chapter: the case of a triangle with an *obtuse* angle will be considered hereafter.

It is usual to denote the lengths of the sides of a triangle opposite to the angles A, B, C respectively by the letters a, b, c.

38. The applications we are about to make of Trigonometry will consist of some examples of the calculation of heights and distances. We shall assume that the lengths of straight lines on the ground can be measured, and also that the angle between any two straight lines which meet at the eye of an observer can be measured. Lengths are usually measured by means of a chain. Angles are usually measured by a sextant or by a theodolite. A sextant will measure the angle between any two straight lines drawn from the observer's eye. A theodolite will measure the angle between any straight line drawn from the observer's eye and the horizontal straight line drawn in the same vertical plane as the former straight line: a theodolite will also measure the angle between two horizontal straight lines drawn from the observer's eye, one in one assigned vertical plane, and the other in another assigned vertical plane. A fuller account of the instruments used in measuring distances will be found in works on Surveying.

39. *To find the distance of an inaccessible point on a horizontal plane.*



Let C be the inaccessible point. Measure any straight line AB in the horizontal plane containing C. At A observe the angle CAB, and at B observe the angle ABC. Then the angle ACB is known, by Euclid I. 32.

Now $$\frac{AC}{AB}=\frac{\sin ABC}{\sin ACB};$$

therefore $$AC=\frac{AB\sin ABC}{\sin ACB}:$$

thus AC is known.

Suppose we require the perpendicular distance CD of C from the straight line AB; we have

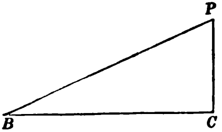
$$\frac{CD}{AC}=\sin CAB;$$

therefore $$CD=AC\sin CAB$$

$$=\frac{AB\sin ABC\sin CAB}{\sin ACB}:$$

thus CD is known.

40. *To find the height of a visible accessible object.*



Let P be the top of the object, and let it be required to find the height PC. Measure any distance CB in a horizontal straight line from the foot of the object; at B observe the angle PBC. Then

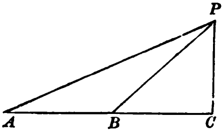
$$\frac{PC}{BC}=\tan PBC,$$

therefore $$PC=BC\tan PBC:$$

thus PC is known.

41. Strictly speaking, in the preceding Article, BC is not a straight line measured *on* the ground, but a straight line *parallel* to the ground at a distance from it equal to the height of the observer's eye at B. Thus PC is the height of the object above the level of the observer's eye; to obtain the height of P measured from the ground we must add to the value of PC the height of the observer's eye at B above the ground. This remark will be applicable to some other Articles in the book; we shall not repeat it, nor need the student supply the correction thus noticed unless it should be definitely required in an example.

42. *To find the height and distance of an inaccessible object on a horizontal plane.*



Let P be the top of an object, and let it be required to find the height PC, and the distance of the object from a given point A in the horizontal plane through C. At A observe the angle PAC, then measure any length AB directly towards the object, and at B observe the angle PBC. Then in the triangle APB the side AB is known, and the angle PAB, and also the angle APB; for the angle APB is the difference of the angles PBC and PAC, by Euclid I. 32.

Now, by Art. 37,

$$\frac{BP}{AB}=\frac{\sin PAB}{\sin APB};$$

therefore $$BP=\frac{AB \sin PAB}{\sin APB};$$

thus BP is known.

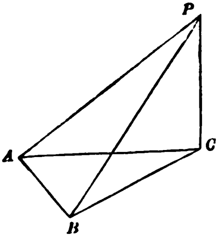
Then $$\frac{PC}{BP}=\sin PBC;$$

therefore $$PC = BP \sin PBC$$

$$= \frac{AB \sin PAB \sin PBC}{\sin APB}:$$

thus PC is known.

43. If however it is not convenient to measure the length AB directly towards the object we may proceed thus: measure the length AB in any direction from A; at A observe the angle PAB, and at B observe the angle PBA, and the angle PBC.



Then in the triangle ABP the side AB is known, and the angles PAB and PBA; and thus the angle APB is known by Euclid I. 32.

Now $$\frac{BP}{AB} = \frac{\sin PAB}{\sin APB};$$

therefore $$BP = \frac{AB \sin PAB}{\sin APB}:$$

thus BP is known.

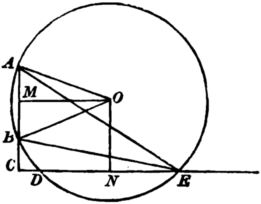
Then $$\frac{PC}{BP} = \sin PBC;$$

therefore $$PC = BP \sin PBC$$

$$= \frac{AB \sin PAB \sin PBC}{\sin APB}:$$

thus PC is known.

44. *An object of known height is situated above a horizontal plane; and the angle subtended by the object at a given point in the plane is observed: it is required to determine the elevation of the object above the plane.*



Let AB be the object of known height; BC the required elevation above the horizontal plane. Suppose a segment of a circle described on AB, containing an angle equal to the given angle subtended by AB: let this circle cut the horizontal plane at D and E.

Let O be the centre of the circle; draw OM perpendicular to AB, and ON perpendicular to DE.

The angle AOM is half of the angle AOB; hence by Euclid III. 20 the angle AOM is equal to the angle AEB, and is therefore known.

And $$\frac{AM}{MO} = \tan AOM;$$

therefore $$MO = AM \cot AOM;$$

and as AM is half of AB we thus determine MO. And

$$CN = MO.$$

Now one of the two distances CD and CE is supposed to be given, namely, the former or the latter according as the given distance is less or greater than MO; and we can determine the other, since CN is known, and N is the middle point of DE.

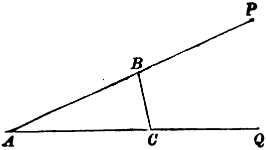
Then by Euclid III. 36, *Corollary*,

$$CD \cdot CE = CA \cdot CB = (CM + MA)(CM - MA)$$
$$= CM^2 - MA^2;$$

therefore $$CM^2 = CD \cdot CE + MA^2.$$

This determines CM, and then CB will be known.

45. It may be remarked that by the aid of measurement of lengths and of calculation we may sometimes avoid the necessity of observing an angle.



Suppose we wish to know the angle PAQ subtended at the point A by straight lines drawn from the points P and Q.

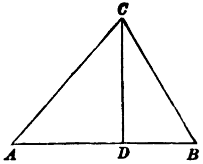
Take any point B in AP; and take C on AQ such that $AC = AB$; and measure BC. Then a perpendicular from A on BC would bisect both the straight line BC and the angle BAC;

therefore $$\sin \tfrac{1}{2} BAC = \frac{\frac{1}{2} BC}{AB}.$$

Since the right-hand side of this equation is known, we can find the angle $\frac{1}{2} BAC$ by the aid of a Table of Sines; see Art. 29. Thus the angle BAC is determined.

46. We will give one example of the use of Trigonometry in Mensuration. We suppose the student to know that the area of a rectangle is measured by the product of the numbers which represent the lengths of two adjacent sides; see the *Notes on the Second Book of Euclid.* The area of a triangle is therefore represented by half the product of its base and altitude, by Euclid I. 41. We shall now be able to demonstrate the proposition of the following Article.

47. *The area of a triangle is equal to half the product of two sides into the sine of the included angle.*



Let ABC be any triangle, CD the perpendicular from C on the base AB.

Then the area $= \frac{1}{2} AB \,.\, CD$;

and $\dfrac{CD}{AC} = \sin BAC$;

therefore $CD = AC \sin BAC$:

thus the area $= \frac{1}{2} AB \,.\, AC \,.\, \sin BAC$.

48. *To express the area of a triangle when one side and the angles are known.*

With the figure of the preceding Article we have, as in Art. 39,

$$AC = \frac{AB \sin ABC}{\sin ACB};$$

therefore the area of the triangle

$$= \frac{AB^2 \,.\, \sin ABC \,.\, \sin BAC}{2 \sin ACB}.$$

49. The area of a parallelogram is double that of a triangle having the same base and altitude: hence by Art. 47 the area of a parallelogram is equal to the product of two adjacent sides into the sine of the included angle.

Similarly we may apply Art. 48 to find the area of a [illegible]llelogram.

EXAMPLES. IV.

1. Given $\tan A = 1{\cdot}05$, find the other Trigonometrical Ratios.

2. Given $\sin A = \dfrac{2n^2 + 2mn}{2n^2 + 2mn + m^2}$, find the other Trigonometrical Ratios.

3. Given $\tan A = \dfrac{pm + qn}{pn - qm}$, find the other Trigonometrical Ratios.

4. Find A from $\sin A + \cos A = \sqrt{\dfrac{3}{2}}$.

5. Find A and B from

$$\tan A \tan B = 1, \quad \tan^2 A + \tan^2 B - \frac{10}{3}.$$

6. Find A and B from

$$\tan A + \tan B = 4, \quad \tan^2 A + \tan^2 B = 14.$$

7. Find A and B from

$$\sin A + \sin B = \sqrt{\frac{3}{2}}, \qquad \sin A \sin B = \frac{1}{4}.$$

8. Shew that the tangents of 60°, 45°, and 15° are in Arithmetical Progression.

9. At a distance of 100 feet from the foot of a tower the tower subtends an angle of 30°; find the height of the tower.

10. A base AB of 100 yards is measured close to the bank of a river, and a tree C on the other bank is observed from A and B; the angle CAB is found to be 60° and the angle CBA is found to be 45°: determine the breadth of the river.

11. A person standing on the bank of a river observes the angle subtended by a tree on the opposite bank to be 75^0, and when he retires 20 feet from the river's edge he finds the angle to be 60^0: determine the height of the tree and the breadth of the river.

12. Find the area of an equilateral triangle, each side being equal to a.

13. Find the area of an isosceles triangle, each of the equal sides being equal to a, and the included angle 30^0.

14. A man 6 feet high standing at the top of a mast subtends an angle whose tangent is $\frac{1}{10}$ at a point on the deck 33 feet from the foot of the mast: find the height of the mast.

15. The upper half of a post, seen from a point on a level with the foot of the post subtends an angle whose tangent is $\frac{1}{3}$: find the tangent of the angle subtended by the whole post.

16. A staff at the top of a tower is observed to subtend an angle of 15^0 by an observer at a distance of a feet from the foot of the tower, and also to subtend the same angle when the observer is at a distance of b feet: find the height of the staff.

17. A column standing on a pedestal 25 feet 6 inches high subtends an angle of 45^0 at the eye of an observer who stands on the horizontal plane from which the pedestal springs. When the observer approaches 20 feet nearer to the column it again subtends an angle of 45^0 at his eye. Find the height of the column supposing the height of the observer's eye above the plane to be 5 feet 6 inches.

18. A person wishing to know the height of a wall, the foot of which was inaccessible, fixed an upright staff 5 feet high, (the height of his eye) at the place where the angular altitude above the level of his eye was 45^0. Having then walked backwards till the angle between the top of the wall and the top of the staff was $18^0 26'$, of which the tangent is $\frac{1}{3}$, he found by actual measurement that his distance from the staff was 70 feet. Determine the height of the wall.

V. *Logarithms.*

50. The numerical calculations which occur in the solution of triangles are abbreviated by the aid of logarithms; and thus it is necessary to explain the nature and the properties of logarithms.

51. Suppose that $a^x = n$, then x is called the *logarithm* of n *to the base* a: thus the logarithm of a number to a given base is the index of the power to which the base must be raised to be equal to the number.

The logarithm of n to the base a is written $\log_a n$: thus if $a^x = n$, then $x = \log_a n$.

52. For example $4^3 = 64$, so that 3 is the logarithm of 64 to the base 4; or $\log_4 64 = 3$.

Again, required the logarithm of 27 to the base 9. Let x denote the required logarithm, so that $9^x = 27$: thus $(3^2)^x = 3^3$, that is $3^{2x} = 3^3$; therefore $2x = 3$, that is $x = 1\frac{1}{2}$.

In the next three Articles we shall give the properties on which the utility of logarithms chiefly depends.

53. *The logarithm of a product is equal to the sum of the logarithms of its factors.*

For let $x = \log_a m$, and $y = \log_a n$;

therefore $m = a^x$, and $n = a^y$;

therefore $mn = a^{x+y}$;

therefore $\log_a mn = x + y = \log_a m + \log_a n$.

54. *The logarithm of a quotient is equal to the logarithm of the dividend diminished by the logarithm of the divisor.*

For let $x = \log_a m$, and $y = \log_a n$;

therefore $m = a^x$, and $n = a^y$;

therefore $\frac{m}{n} = \frac{a^x}{a^y} = a^{x-y}$;

therefore $\log_a \frac{m}{n} = x - y = \log_a m - \log_a n$.

55. *The logarithm of any power, integral or fractional, of a number is equal to the product of the logarithm of the number by the index of the power.*

For let $m = a^x$; therefore $m^r = (a^x)^r = a^{xr}$,

therefore $\log_a (m^r) = xr = r \log_a m$.

56. *To find the relation between the logarithms of the same number to different bases.*

Let $x = \log_a m$, and $y = \log_b m$;

therefore $m = a^x$ and $= b^y$;

therefore $a^x = b^y$;

therefore $a^{\frac{x}{y}} = b$, and $b^{\frac{y}{x}} = a$;

therefore $\frac{x}{y} = \log_a b$, and $\frac{y}{x} = \log_b a$.

Hence $y = x \log_b a$, and $= \frac{x}{\log_a b}$.

Hence the logarithm of a number to the base b may be found by multiplying the logarithm of the number to the base a by $\log_b a$ or by $\frac{1}{\log_a b}$.

Since $\log_b a = \frac{1}{\log_a b}$ we have

$$\log_b a \times \log_a b = 1.$$

57. There are two systems of logarithms which are used in Mathematics.

In one system the base is a certain number which cannot be expressed exactly; as far as nine places of decimals the number is 2·718281828.

This number is usually denoted by the letter e; and logarithms to this base are called *Napierian logarithms*, from Napier the inventor of logarithms. This system of logarithms, although very important in theory, is not used in practical calculations; and we shall not require to consider it in the present work.

In the other system the base is 10; this system is used in practical calculations, and is called the *common* system.

58. We shall not in the present work explain how a table of logarithms is calculated; for this the student may refer to the larger treatise. We may remark that in very few cases can a logarithm be assigned *exactly*, but as close an approximate value as we please can be found; for example, a table may be constructed which shall give logarithms to seven places of decimals.

We shall shew in the next three Articles what are the chief advantages of the common system of logarithms.

59. *In the common system of logarithms if the logarithm of any number be known, we can immediately determine the logarithm of the product or quotient of that number by any power of* 10.

For $\log_{10} N \times 10^n = \log_{10} N + \log_{10} 10^n = \log_{10} N + n$;

$$\log_{10} \frac{N}{10^n} = \log_{10} N - \log_{10} 10^n = \log_{10} N - n.$$

That is, if we know the logarithm of any number we can determine the logarithm of any number which has the same figures, but differs merely by the position of the decimal point.

In future we shall for brevity use *log* for *log*$_{10}$, that is we shall omit to specify the base 10.

60. We know from Arithmetic that

$$10^0 = 1,\ 10^1 = 10,\ 10^2 = 100,\ 10^3 = 1000, \ldots$$

Now from this we infer that if a number lies between 1 and 10, its logarithm lies between 0 and 1; if a number lies between 10 and 100, its logarithm lies between 1 and 2; if a number lies between 100 and 1000 its logarithm lies between 2 and 3: and so on.

The integral part of a logarithm is called the *characteristic*, and the decimal part is called the *mantissa;* thus as the logarithm of any number between 100 and 1000 is greater than 2 and less than 3, it is equal to 2+some decimal: thus in this case 2 is the characteristic.

We shall now give an important proposition respecting the characteristic.

61. *In the common system of logarithms the characteristic of the logarithm of any number can be determined by inspection.*

For suppose the number to be greater than unity, and to lie between 10^n and 10^{n+1}; then the logarithm is greater than n and less than $n+1$, so that the characteristic of the logarithm is n. Next suppose the number to be less than unity, and to lie between $\frac{1}{10^n}$ and $\frac{1}{10^{n+1}}$, that is between 10^{-n} and 10^{-n-1}; then the logarithm will be some negative quantity between $-n$ and $-(n+1)$; hence if we agree that the *mantissa shall always be positive*, the characteristic of the logarithm will be $-(n+1)$.

Hence we have the following rule: the characteristic of the logarithm of a number is *one less* than the number of integral figures of the number; when the number has no integral figures the characteristic of the logarithm is negative and is *one more* than the number of cyphers immediately to the right of the decimal place in the number.

62. By reason of the properties explained in the three preceding Articles it is unnecessary in a table of common logarithms to print either the characteristics of the logarithms or the decimal points of the numbers.

For example, we find in a table the following figures:

Number.	Logarithm.
15627	1938756

This means that ·1938756 is the *mantissa;* for the number 15627 the corresponding characteristic is 4, and therefore

$$\log 15627 = 4{\cdot}1938756.$$

Similarly $\log 156{\cdot}27 = 2{\cdot}1938756$, and $\log {\cdot}0015627 = \bar{3}{\cdot}1938756$: in the last example $\bar{3}$ is equivalent to -3, so that we express in the manner indicated the fact that

$$\log {\cdot}0015627 = -3 + {\cdot}1938756.$$

63. It is necessary to notice one point in practical operations with negative characteristics.

Suppose we require the logarithm of the cube root of ·0015627. By Art. 55 the logarithm is $\frac{1}{3}$ of $\bar{3}{\cdot}1938756$. The division here can be immediately effected; for $\frac{1}{3}$ of -3 is -1; and $\frac{1}{3}$ of ·1938756 is ·0646252: thus the required logarithm is $\bar{1}{\cdot}0646252$.

But suppose we require the logarithm of the square root of ·0015627. By Art. 55 the logarithm is $\frac{1}{2}$ of $\bar{3}{\cdot}1938756$. It is convenient now to put $\bar{3}{\cdot}1938756$ in the form $-4+1{\cdot}1938756$; then dividing by 2 we obtain $-2+{\cdot}5969378$, so that the required logarithm is $\bar{2}{\cdot}5969378$.

Similarly if we require the logarithm of the sixth root of ·0015627 we put $\bar{3}{\cdot}1938756$ in the form $-6+3{\cdot}1938756$; then dividing by 6 we obtain $-1+{\cdot}5323126$, so that the required logarithm is $\bar{1}{\cdot}5323126$.

64. The following example will illustrate the present Chapter.

Given $\log 3={\cdot}4771213$ find the log of $\dfrac{(2{\cdot}7)^3\times({\cdot}81)^{\frac{4}{5}}}{(90)^{\frac{5}{4}}}$.

Let N denote the given expression; then

$$\begin{aligned}\log N&=\log(2{\cdot}7)^3+\log({\cdot}81)^{\frac{4}{5}}-\log(90)^{\frac{5}{4}}\\&=3\log 2{\cdot}7+\frac{4}{5}\log{\cdot}81-\frac{5}{4}\log 90.\end{aligned}$$

$$\text{Now }\log 2{\cdot}7=\log\frac{27}{10}=\log\frac{3^3}{10}=3\log 3-1,$$

$$\log{\cdot}81=\log\frac{81}{100}=\log\frac{3^4}{10^2}=4\log 3-2,$$

$$\log 90=\log 3^2\times 10=2\log 3+1;$$

hence $\log N=$

$$\begin{aligned}&3(3\log 3-1)+\frac{4}{5}(4\log 3-2)-\frac{5}{4}(2\log 3+1)\\&=\left(9+\frac{16}{5}-\frac{5}{2}\right)\log 3-3-\frac{8}{5}-\frac{5}{4}\\&=\frac{97}{10}\log 3-\frac{117}{20}=\bar{2}{\cdot}7780766\text{ nearly.}\end{aligned}$$

EXAMPLES. V.

Find the logarithms of the following six numbers to the assigned bases:

1. 256 to the base 2. 2. 32 to the base 4.
3. 243 to the base 9. 4. 16 to the base 8.
5. 64 to the base 16. 6. 128 to the base 32.

Given log 2 = ·3010300, log 3 = ·4771213, find the logarithms of the following twelve numbers:

7. 18. 8. 60. 9. 216. 10. 6480.
11. 5400. 12. $\frac{4}{9}$. 13. 4·32. 14. ·72.
15. ·375. 16. ·03. 17. $6^{-\frac{1}{3}}$. 18. $(5\frac{1}{3})^{-\frac{1}{2}}$.

Given log 3 = ·4771213, log 7 = ·8450980, find the logarithms of the following three numbers:

19. 63. 20. $\frac{9}{49}$. 21. $\frac{1}{441}$.

Given log 8 = ·9030900, log 9 = ·9542425, find the logarithms of the following three numbers:

22. $1\frac{1}{8}$. 23. $\sqrt{1\frac{1}{2}}$. 24. $\sqrt[3]{1\frac{1}{8}}$.

25. Given log 8 = ·9030900, log 27 = 1·4313638, find log $2\frac{1}{4}$.

26. Write down the characteristics of the logarithms of 3·4512, 345·12, ·034512, and ·000034512: also having given log ·34512 = $\bar{1}$·5379701, find the logarithm of the product of the above four numbers.

27. The decimal part of the log of 36541 is ·5627804, find the log of $\sqrt[15]{(\cdot 000036541)}$.

28. Find the log of ·0625 to the base 8.

29. Given log 1·4 = ·1461280, log 1·5 = ·1760913, find log ·000315.

30. Given log 2 find the log of 50 to the base 25.

31. Given log 2 find the log of 1000 to the base 25.

32. Given log 2 and log 3 find x from $(1\cdot 08)^x = 1000$.

VI. *Use of Tables.*

65. Many collections of Mathematical Tables have been published, differing in extent and in the number of decimal places to which they are carried; and thus practical calculators are enabled to provide themselves with such Tables as are most convenient for the special work on which they may be engaged. A collection of Tables published by W. and R. Chambers may suffice for ordinary purposes. A cheap and very extensive collection of Tables has been edited in Germany by Schrön, and this work has been introduced into England with a Preface by Professor De Morgan.

66. Collections of Tables usually contain explanations of the mode in which they are arranged, together with instructions for using them. We shall accordingly only give here some examples which will suffice to guide the student who may wish to use any Tables for occasional calculation. We shall not give investigations of the accuracy of the methods which we exemplify; for such investigations the student is referred to the larger treatise.

67. One general consideration which applies to the use of Mathematical Tables is this: we rarely find what we require immediately in the Tables, but we find two entries between which what we require must lie, and from which it must be determined. Accordingly we have to exemplify the method of proceeding in such cases.

68. *To find the logarithm of a given number.*

If the given number is contained in the Table we take the decimal part of the logarithm from the Table, and prefix the characteristic; see Art. 61.

Suppose however that the number is not contained exactly in the Table. The Table, for example, may give the logarithms of all numbers from 1 to 100000, and we may require the logarithm of 5632147.

Here we take from the table

Number.	Logarithm.
56321	7506704
56322	7506781.

Hence, by Art. 61,

$$\log 5632100 = 6\cdot7506704,$$
$$\log 5632200 = 6\cdot7506781.$$

The difference of the two numbers is 100, and the difference of the two logarithms is ·0000077. Let x denote the quantity to be added to the logarithm of 5632100, in order to produce the logarithm of 5632147; then *we assume* that

$$100 : 47 :: \cdot0000077 : x;$$

that is, we assume that for a small change in the number there is a *proportional* small change in the logarithm.

Hence we obtain $x = \frac{47}{100} \times \cdot0000077$, that is ·000003619, or to seven places of decimals ·0000036.

And $\cdot7506704 + \cdot0000036 = \cdot7506740$.

Therefore $\log 5632147 = 6\cdot7506740$.

Then, by Art. 61, we can immediately express the logarithm of any other number which is formed from 5632147 by supplying a decimal point; for example

$$\log 56321\cdot47 = 4\cdot7506740.$$

69. *To find the number corresponding to a given logarithm.*

If the decimal part of the given logarithm is contained in the Table we take the corresponding number, and put a decimal point in the number in the place indicated by the given characteristic.

Suppose however that the decimal part of the logarithm is not contained exactly in the Table; we shall then have to perform a process like that exemplified in the preceding Article. For example, suppose the given logarithm to be 2·7506740.

As before we have

$$\log 5632100 = 6\cdot7506704,$$
$$\log 5632200 = 6\cdot7506781.$$

Let x denote the quantity to be added to 5632100 to produce the number which has 6·7506740 for its logarithm.

Since $6{\cdot}7506781 - 6{\cdot}7506704 = {\cdot}0000077$,

and $6{\cdot}7506740 - 6{\cdot}7506704 = {\cdot}0000036$,

we form the proportion

$${\cdot}0000077 : {\cdot}0000036 :: 100 : x.$$

Hence $x = \dfrac{3600}{77} = 47$ approximately,

therefore $\log 5632147 = 6{\cdot}7506740$,

and therefore $\log 563{\cdot}2147 = 2{\cdot}7506740$.

Thus the required number is 563·2147.

70. In using *Trigonometrical* Tables processes have to be performed like those exemplified in the two preceding Articles for Tables of *Logarithms*. For example, we may have a Table of the sines of those angles which are expressible exactly in degrees and minutes, and we may require the sine of such an angle as $20^0\,14'\,20''$: in this case we must proceed as in Art. 68.

We take from the Table

$$\sin 20^0\,14' = {\cdot}3458441,$$
$$\sin 20^0\,15' = {\cdot}3461171.$$

Let x denote the quantity to be added to ·3458441 to produce the sine of $20^0\,14'\,20''$.

Since ${\cdot}3461171 - {\cdot}3458441 = {\cdot}0002730$,

we form the proportion

$$60'' : 20'' :: {\cdot}0002730 : x.$$

Hence $x = \dfrac{20}{60} \times {\cdot}0002730 = {\cdot}0000910$;

therefore $\sin 20^0\,14'\,20'' = {\cdot}3458441 + {\cdot}0000910 = {\cdot}3459351$

71. Tables such as those referred to in the preceding Article are called Tables of *Natural* sines, cosines, tangents,... to distinguish them from other Tables which are called Tables of *Logarithmic* sines, cosines, tangents,... We shall now consider the latter kind of Tables.

72. The Trigonometrical Ratios of an angle are numerical quantities, and it is found very convenient to have Tables which give the *logarithms* of these numerical quantities, so that we may be saved the trouble of calculating them by the aid of the Table of Logarithms. For abbreviation log sin A is used to denote the logarithm of the sine of A; and in a similar manner log cos A, log tan A,... are used.

73. Since the sine of an angle is never greater than unity the logarithm of a sine will never be a positive quantity; the same remark applies to the cosine. In order to avoid the occurrence of negative quantities in the Tables it is found convenient to *add* 10 *to the logarithm of every Trigonometrical Ratio* before registering it in the Tables; the logarithm so increased is called the *Tabular logarithm*, and is usually denoted by the letter L. Thus

$$L \sin A = \log \sin A + 10,$$
$$L \tan A = \log \tan A + 10,$$

and so on. Of course in calculation we shall have to remember and allow for this addition to the real logarithms of the Trigonometrical Ratios.

74. There is one point to which *special attention must be paid* in using both the Tables of the natural Trigonometrical Ratios and the Tables of the logarithmic Trigonometrical Ratios, namely, that as the angle *increases* the sine, tangent, and secant *increase*, but the cosine, cotangent and cosecant *decrease*; the bearing of this remark will be illustrated in the next Article.

75. We will now give some examples of the use of the Tables of the logarithmic Trigonometrical Ratios.

Given $L \sin 20^\circ\, 14' = 9{\cdot}5388804,$

$L \sin 20^\circ\, 15' = 9{\cdot}5392230;$

required $L \sin 20^\circ\, 14'\, 20''.$

The difference of the given Tabular logarithms is ·0003426, which corresponds to the difference 60″ in the angles; so we form the proportion

$$60 : 20 :: {\cdot}0003426 : x.$$

Hence $x = \frac{20}{60} \times {}^{\cdot}0003426 = {}^{\cdot}0001142$;

therefore $L \sin 20^0\,14'\,20'' = 9{\cdot}5388804 + {}^{\cdot}0001142$
$= 9{\cdot}5389946$.

The same data will furnish an example of the calculation of a logarithmic cosine.

Given $L \cos 69^0\,45' = 9{\cdot}5392230$,
$L \cos 69^0\,46' = 9{\cdot}5388804$;
required $L \cos 69^0\,45'\,40''$.

Here the proportion is

$$60 : 40 :: {}^{\cdot}0003426 : x.$$

Hence $x = \frac{40}{60} \times {}^{\cdot}0003426 = {}^{\cdot}0002284$;

therefore $L \cos 69^0\,45'\,40'' = 9{\cdot}5392230 - {}^{\cdot}0002284$
$= 9{\cdot}5389946$.

Here x is *subtracted* from $L \cos 69^0\,45'$ because as the angle increases, the cosine decreases, and so also does the L cosine.

The two preceding examples resemble that in Art. 68; we will now take one resembling that in Art. 69.

Given $L \sin 20^0\,14' = 9{\cdot}5388804$,
$L \sin 20^0\,15' = 9{\cdot}5392230$;
find the angle which has for its L sine $9{\cdot}5389946$.

Let x denote the required number of seconds.

Since $9{\cdot}5392230 - 9{\cdot}5388804 = {}^{\cdot}0003426$,
and $9{\cdot}5389946 - 9{\cdot}5388804 = {}^{\cdot}0001142$,
we form the proportion

$${}^{\cdot}0003426 : {}^{\cdot}0001142 :: 60 : x.$$

Hence $x = 60 \times \frac{{}^{\cdot}0001142}{{}^{\cdot}0003426} = 20$.

Hence the required angle is $20^0\,14'\,20''$.

EXAMPLES. VI.

1. Given log 34586 = 4·5389003, log 34587 = 4·5389129, find log 345·8637.

2. Given log 41501 = 4·6180586, log 41502 = 4·6180690, find log 4150145.

3. Given log 7·3510 = ·8663464, log 7·3511 = ·8663523, find log 735·1092.

4. Given log 1752 = 3·2435341, log 1752·1 = 3·2435589, find log 17·52087.

5. Given log 6·1025 = ·7855078, log 6·1026 = ·7855149, find log 610·257.

6. Given log 61875 = 4·7915152, log 61876 = 4·7915222, find log 6187539.

7. Given log 61601 = 4·7895878, log 61602 = 4·7895948, find the number corresponding to the logarithm 2·7895912.

8. Given log 7·5014 = ·8751423, log 7·5015 = ·8751481, find the number corresponding to the logarithm $\bar{3}\cdot8751462$.

9. Given log 1·3107 = ·1175033, log 131·08 = 2·1175364, log 5 = ·6989700, find the seventeenth root of 131072.

10. Find $(1\cdot05)^{20}$ having given

    log 2 = ·3010300, log 2·653 = ·4237372,
    log 3 = ·4771213, log 2·654 = ·4239009,
    log 7 = ·8450980.

11. Find $L \sin 38^\circ\, 24'\, 27''$, having given

    $L \sin 38^\circ\, 24' = 9\cdot7931949$,
    $L \sin 38^\circ\, 25' = 9\cdot7933543$.

12. Find $L \sin 32^\circ\, 28'\, 36''$, having given

    $L \sin 32^\circ\, 28' = 9\cdot7298197$
    $L \sin 32^\circ\, 29' = 9\cdot7300182$.

13. Find $L \sin 41^\circ\, 50'\, 34''\cdot5$, having given

    $L \sin 41^\circ\, 50'\, 30'' = 9\cdot8241743$,
    $L \sin 41^\circ\, 50'\, 40'' = 9\cdot8241978$.

14. Find $L\cos 17^\circ 31' 25''{\cdot}2$, having given
$L\cos 17^\circ 31' = 9{\cdot}9793796$,
$L\cos 17^\circ 32' = 9{\cdot}9793398$.

15. Find $L\tan 21^\circ 17' 12''$, having given
$L\tan 21^\circ 17' = 9{\cdot}5905617$,
$L\tan 21^\circ 18' = 9{\cdot}5909351$.

16. Find $L\tan 27^\circ 26' 42''$, having given
$L\tan 27^\circ 26' = 9{\cdot}7152419$,
$L\tan 27^\circ 27' = 9{\cdot}7155508$.

17. Find $L\tan 55^\circ 37' 53''$, having given
$L\tan 55^\circ 37' = 10{\cdot}1647616$,
diff. for $1' = {\cdot}0002711$.

18. Find $L\,\text{cosec}\, 33^\circ 10' 20''$, having given
$L\sin 33^\circ 10' = 9{\cdot}7380479$,
$L\sin 33^\circ 11' = 9{\cdot}7382412$.

19. Given $L\sin 16^\circ = 9{\cdot}4403381$, diff. for $1' = {\cdot}0004403$,
$L\cos 16^\circ = 9{\cdot}9828416$, diff. for $1' = {\cdot}0000362$,
find $L\sec 16^\circ 0' 27''$ and $L\tan 16^\circ 0' 27''$.

20. Find A, having given
$L\sin A = 9{\cdot}4488105$,
$L\sin 16^\circ 19' = 9{\cdot}4486227$,
$L\sin 16^\circ 20' = 9{\cdot}4490540$.

21. Find A, having given
$L\sin A = 9{\cdot}0787743$,
$L\sin 6^\circ 53' = 9{\cdot}0786310$,
$L\sin 6^\circ 53' 10'' = 9{\cdot}0788054$.

22. Find A, having given
$L\cos A = 9{\cdot}9657056$,
$L\cos 22^\circ 28' 20' = 9{\cdot}9657025$,
$L\cos 22^\circ 28' 10'' = 9{\cdot}9657112$.

23. Find A, having given
$L\cos A = 9{\cdot}2000000$,
$L\cos 80^\circ 53' = 9{\cdot}1998793$,
$L\cos 80^\circ 52' 50'' = 9{\cdot}2000105$.

24. If $\log x = \sqrt[3]{10}$ find x, having given
$\log 21544 = 4{\cdot}3333263$, $\log 14270 = 4{\cdot}1544240$,
$\log 21545 = 4{\cdot}3333465$, $\log 14271 = 4{\cdot}1544544$.

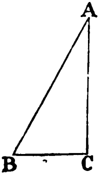
VII. *Solution of Right-Angled Triangles.*

76. In every triangle there are six elements, namely the three sides and the three angles. When we have a sufficient number of these elements given we can calculate the remaining elements; this process is called the *solution of triangles.* It will appear as we proceed that when three of the elements are given we can calculate the remaining three, except when the three angles are given, and then we cannot determine the three sides but only the *ratio* they bear to each other.

We have already in Chapter IV. given some examples of the solution of triangles, and in the course of the present Chapter and a future Chapter we shall examine every case which can occur. It is usual to consider separately the case of right-angled triangles as the investigations are more simple for these than for other triangles; accordingly we shall confine ourselves in the present Chapter to right-angled triangles.

77. We shall use the notation given in Art. 37 for the sides of a triangle; and we shall always suppose that C is the right angle in a right-angled triangle. We use *log* as an abbreviation for logarithm, and we use L in the sense explained in Art. 73.

78. *To solve a right-angled triangle having given the hypotenuse and an acute angle.*



Suppose the hypotenuse and the angle A given; then

$$B = 90^\circ - A;$$

$$\frac{a}{c} = \sin A, \text{ therefore } a = c \sin A,$$

fore $\log a = \log c + \log \sin A = \log c + L \sin A - 10;$

$$\frac{b}{c} = \sin B, \text{ therefore } b = c \sin B,$$

therefore $\log b = \log c + \log \sin B = \log c + L \sin B - 10.$

Thus B, a, and b are determined.

79. *To solve a right-angled triangle having given the hypotenuse and a side.*

Suppose c and a given; then

$$\sin A = \frac{a}{c}, \ \log \sin A = \log a - \log c,$$

therefore $$L \sin A = 10 + \log a - \log c$$

this determines A; then $B = 90^\circ - A$.

And $c^2 = a^2 + b^2$, therefore $b^2 = c^2 - a^2 = (c-a)(c+a)$,

therefore $$b = \sqrt{(c-a)(c+a)},$$

$$\log b = \tfrac{1}{2} \log (c-a) + \tfrac{1}{2} \log (c+a).$$

Or we may find b from the formula

$$b = c \cos A.$$

80. *To solve a right-angled triangle having given a side and an acute angle.*

Suppose a and A given; then

$$B = 90^\circ - A;$$

$$\frac{a}{c} = \sin A, \text{ therefore } c = \frac{a}{\sin A},$$

therefore $\log c = \log a - \log \sin A = \log a - L \sin A + 10;$

$$\frac{a}{b} = \tan A, \text{ therefore } b = \frac{a}{\tan A},$$

therefore $\log b = \log a - \log \tan A = \log a - L \tan A + 10.$

Thus B, c, b are determined.

Suppose a and B are given; then $A = 90^\circ - B$, and we may find c and b as before.

81. *To solve a right-angled triangle having given the two sides.*

Here a and b are given; then

$$\tan A = \frac{a}{b}, \text{ therefore } \log \tan A = \log a - \log b,$$

therefore $$L \tan A = 10 + \log a - \log b;$$

$$B = 90^0 - A;$$

$$\frac{a}{c} = \sin A, \text{ therefore } c = \frac{a}{\sin A};$$

therefore $$\log c = \log a - \log \sin A = \log a - L \sin A + 10.$$

Or we may find c from the formula $c = \sqrt{(a^2 + b^2)}$, but this is not adapted to logarithmic computation.

82. It will thus be seen that in each of the four cases discussed in Arts. 78 to 81 we suppose that we know two elements of a right-angled triangle, besides the right angle, and we shew how the other elements are to be determined. And it will be found on examination that we have discussed every case in which two elements are given besides the right angle, except the case in which the two angles are given. In this case we can determine the *ratio* of each side to the hypotenuse; for we have

$$\frac{a}{c} = \sin A, \qquad \frac{b}{c} = \sin B;$$

but we cannot absolutely determine a, b, and c. We may observe that since $A + B = 90^0$ it is superfluous to give the values of both A and B, for if one is given the other can be immediately found.

83. We will now take some examples of the solution of right-angled triangles.

Ex. (1). Suppose we have given $c = 125$, $A = 54^0 28'$.

$$a = c \sin A.$$

Using a table of natural sines we find

$$\sin 54^0 28' = \cdot 8137775;$$

therefore $$a = 101 \cdot 7221875.$$

Or, using logarithms,

$$\begin{array}{r} \log 125 = 2{\cdot}0969100 \\ L\sin 54^\circ\, 28' = \underline{9{\cdot}9105057} \\ 12{\cdot}0074157 \end{array}$$

therefore $\log a = 2{\cdot}0074157$.

Now on consulting the tables of logarithms we find

$$\log 101{\cdot}73 = 2{\cdot}0074490,$$

$$\log 101{\cdot}72 = 2{\cdot}0074064;$$

and as $\log a$ lies between the logarithms here given we conclude that a lies between 101·72 and 101·73.

In order to determine a more closely we must employ the *Principle of proportional parts* which is explained in Chapter VI. We will give the process for the present case.

$$\begin{array}{rr} & 2{\cdot}0074490 \\ & \underline{2{\cdot}0074064} \\ \text{Diff.} & {\cdot}0000426 \end{array} \qquad \begin{array}{rr} & 2{\cdot}0074157 \\ & \underline{2{\cdot}0074064} \\ \text{Diff.} & {\cdot}0000093 \end{array}$$

Let x denote the excess of a above 101·72; then

$${\cdot}0000426 : {\cdot}0000093 :: {\cdot}01 : x.$$

Hence we find $x = {\cdot}00218$ nearly.

Thus $\quad a = 101{\cdot}72 + {\cdot}00218 = 101{\cdot}72218.$

Our two modes of calculation give values for a which differ very slightly.

$$B = 90^\circ - A = 35^\circ\, 32',$$

$$b = c \sin B.$$

Using a table of natural sines we find

$$\sin 35^\circ\, 32' = {\cdot}5811765,$$

$$b = 72{\cdot}6470625.$$

Or, using logarithms,

$$\begin{array}{r} \log 125 = 2{\cdot}0969100 \\ L\sin 35^\circ\, 32' = \underline{9{\cdot}7643080} \\ 11{\cdot}8612180 \end{array}$$

Therefore $\quad \log b = 1{\cdot}8612180.$

Now on consulting the tables of logarithms we find

$$\log 72{\cdot}648 = 1{\cdot}8612237,$$

$$\log 72{\cdot}647 = 1{\cdot}8612177.$$

Hence b is very nearly equal to 72·647.

Ex. (2). Suppose we have given $a = 147$, $c = 184$.

$$L \sin A = 10 + \log a - \log c.$$

$$\log 147 = 2{\cdot}1673173,$$

$$\log 184 = 2{\cdot}2648178,$$

therefore $$L \sin A = 9{\cdot}9024995.$$

Now on consulting the tables we find

$$L \sin 53^{0}\, 2' = 9{\cdot}9025389,$$

$$L \sin 53^{0}\, 1' = 9{\cdot}9024438;$$

and we conclude that A lies between $53^{0}\, 1'$ and $53^{0}\, 2'$. Let x denote the number of seconds in the excess of A above $53^{0}\, 1'$. Proceed as before,

$$\begin{array}{rr} & 9{\cdot}9025389 \\ & \underline{9{\cdot}9024438} \\ \text{Diff.} & {\cdot}0000951 \end{array} \qquad \begin{array}{rr} & 9{\cdot}9024995 \\ & \underline{9{\cdot}9024438} \\ \text{Diff.} & {\cdot}0000557. \end{array}$$

$${\cdot}0000951 : {\cdot}0000557 :: 60 : x.$$

Thus $x = 35$ nearly,

and $A = 53^{0}\, 1'\, 35''$ nearly,

$$B = 90^{0} - A = 36^{0}\, 58'\, 25''.$$

If we had only a table of natural sines we should proceed thus:

$$\sin A = \frac{a}{c} = \frac{147}{184} = {\cdot}7989130 \text{ nearly.}$$

On consulting the tables we find

$$\sin 53^{0}\, 2' = {\cdot}7989855,$$

$$\sin 53^{0}\, 1' = {\cdot}7988105;$$

and we conclude that A lies between $53^{0}\, 1'$ and $53^{0}\, 2'$. Then we may proceed as before to determine A more closely; and we shall obtain $A = 53^{0}\, 1'\, 35''$ nearly.

And $b^2 = c^2 - a^2 = (c-a)(c+a) = 37 \times 331$,

$$\begin{aligned} \log 37 &= 1{\cdot}5682017 \\ \log 331 &= 2{\cdot}5198280 \\ 2\,&\underline{\big)\,4{\cdot}0880297} \\ \log b &= 2{\cdot}0440148 \end{aligned}$$

Now on consulting the tables of logarithms we find

$$\log 110{\cdot}67 = 2{\cdot}0440299,$$

$$\log 110{\cdot}66 = 2{\cdot}0439907;$$

and hence b lies between 110·66 and 110·67.

Let x denote the excess of b above 110·66. Proceed as before;

$$\begin{array}{rr} 2{\cdot}0440299 & 2{\cdot}0440148 \\ \underline{2{\cdot}0439907} & \underline{2{\cdot}0439907} \\ \text{Diff. } {\cdot}0000392 & \text{Diff. } {\cdot}0000241 \end{array}$$

$${\cdot}0000392 : {\cdot}0000241 :: {\cdot}01 : x.$$

Thus $x = {\cdot}00615$ nearly;

and $b = 110{\cdot}66615$ nearly.

We might also obtain b by extracting the square root of 37×331; without using logarithms.

Ex. (3). Suppose we have given $a = 237{\cdot}6$, $A = 34^\circ\ 18'$.

$$\log c = \log a - L \sin A + 10,$$

$$\begin{aligned} \log 237{\cdot}6 &= 2{\cdot}3758464 \\ L \sin 34^\circ\ 18' &= \underline{9{\cdot}7509140} \\ \log c &= 2{\cdot}6249324 \end{aligned}$$

Now on consulting the tables of logarithms we find

$$\log 421{\cdot}64 = 2{\cdot}6249418,$$

$$\log 421{\cdot}63 = 2{\cdot}6249315;$$

and hence c lies between 421·63 and 421·64.

Let x denote the excess of c above 421·63. Proceed as before;

$$\begin{array}{rr@{\qquad}rr} & 2{\cdot}6249418 & & 2{\cdot}6249324 \\ & \underline{2{\cdot}6249315} & & \underline{2{\cdot}6249315} \\ \text{Diff.} & {\cdot}0000103 & \text{Diff.} & {\cdot}0000009 \end{array}$$

$${\cdot}0000103 : {\cdot}0000009 :: {\cdot}01 : x.$$

Thus $\qquad x = {\cdot}0009$ nearly;

and $\qquad c = 421{\cdot}6309$ nearly.

$$\log b = \log a - L \tan A + 10,$$

$$\begin{array}{r} \log 237{\cdot}6 = 2{\cdot}3758464 \\ L \tan 34^{0}\,18' = \underline{9{\cdot}8338823} \\ \log b = 2{\cdot}5419641 \end{array}$$

Now on consulting the tables of logarithms we find

$$\log 348{\cdot}31 = 2{\cdot}5419659,$$

$$\log 348{\cdot}30 = 2{\cdot}5419535;$$

and hence b lies between 348·30 and 348·31.

Let x denote the excess of b above 348·30. Proceed as before,

$$\begin{array}{r@{\qquad}r} 2{\cdot}5419659 & 2{\cdot}5419641 \\ \underline{2{\cdot}5419535} & \underline{2{\cdot}5419535} \\ {\cdot}0000124 & {\cdot}0000106 \end{array}$$

$${\cdot}0000124 : {\cdot}0000106 :: {\cdot}01 : x.$$

Thus $\qquad x = {\cdot}00855$ nearly;

and $\qquad b = 348{\cdot}30855$ nearly.

84. The process of applying the principle of proportional parts is in practice much simplified by the aid of the Tables; we find in fact that the chief part of the calculation is performed for us. This will be obvious to the student on examining the Tables and the explanations which usually accompany them.

EXAMPLES. VII.

Solve the following eight triangles from the given quantities:

1. $c=150$, $A=30^0$, $C=90^0$.

2. $c=200$, $a=100$, $C=90^0$.

3. $a=80$, $B=15^0$, $C=90^0$.

4. $a=75$, $b=75$, $C=90^0$.

5. $c=120$, $B=36^0$, $C=90^0$.
   $\sin 36^0 = \cdot 5877853$, $\sin 54^0 = \cdot 8090170$.

6. $c=290$, $a=200$, $C=90^0$.
   $\sin 43^0\, 36' = \cdot 6896195$, $\sin 43^0\, 37' = \cdot 6898302$.

7. $a=125$, $B=22\frac{1}{2}^0$, $C=90^0$.

8. $a=3$, $b=4$, $C=90^0$.
   $\log 2 = \cdot 3010300$, $\sin 53^0\, 7' = 9\cdot 9030136$,
   $L \sin 53^0\, 8' = 9\cdot 9031084$.

In solving the following four triangles the Tables will be required:

9. $c=196$, $A=23^0\, 30'$, $C=90^0$.

10. $c=164$, $a=96$, $C=90^0$.

11. $a=124\cdot 6$, $A=64^0\, 20'$, $C=90^0$.

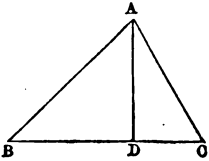
12. $a=141$, $b=193$, $C=90^0$.

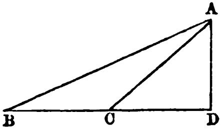
VIII. *Solution of oblique-angled triangles by the aid of right-angled triangles.*

85. In the preceding Chapters we have restricted ourselves to the Trigonometrical Ratios of angles not greater than a right angle; and we have in effect given a short course of Trigonometry as far as the solution of right-angled triangles inclusive. It is however obvious that we may have triangles with obtuse angles, and this leads us to extend our definitions of the Trigonometrical Ratios so as to include angles greater than a right angle. Accordingly we shall devote the next Chapter to this subject; and then we shall proceed in the following Chapters to explain certain properties of triangles and the general solution of triangles.

But it may be convenient for some students to be able to solve any triangle without entering on the consideration of the Trigonometrical Ratios of angles greater than a right angle; and the present Chapter will supply the necessary rules and explanations. Those who adopt the methods of solution to be given in Chapter XI. may look on the present Chapter as an application and illustration of the elementary formulæ of the subject.

86. *To solve a triangle having given two angles and a side.*





Suppose c the given side; since two angles are given all the angles are known. Draw AD perpendicular to BC or to BC produced.

In the right-angled triangle *ABD* we know the hypotenuse *AB*, and the angle *ABD*; hence we can find *AD* and *BD*, by Art. 78.

Then in the right-angled triangle *ADC* we know *AD* and the angle *ACD*; hence we can find *AC* and *CD*, by Art. 80.

And, knowing *BD* and *CD*, we find *BC* immediately.

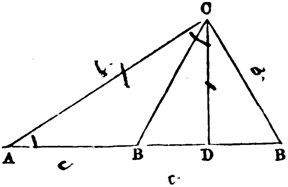
87. *To solve a triangle having given two sides and the included angle.*

The solution is given in Arts. 105 and 110, and as it requires no principles which have not been already explained it may be read at this stage.

88. *To solve a triangle having given two sides and the angle opposite to one of them.*

Let *a* and *b* be the given sides, and *A* the given angle.

I. Suppose *a* less than *b*.



Then *A* is less than *B*, and so *A* must be an acute angle.

Let *AC* denote the side *b*, and from *C* draw the perpendicular *CD* on the opposite side of the triangle, produced if necessary.

In the right-angled triangle *ACD* we know *AC* and the angle *A*; hence we can find *CD*, and the angle *ACD*, by Art. 78.

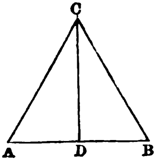
Then in the right-angled triangle *CBD* we know *CD* and *CB*; hence we can find *BD* and the angle *BCD*, by Art. 79.

And thus *AB* and the angle *ACB* may be found immediately.

It will be seen that there are *two* triangles ABC, one having the angle ABC *obtuse*, and one having the angle ABC *acute*. Hence this is usually called the *ambiguous case*, because corresponding to the given elements *two* triangles may *generally* be found.

We say that two triangles may *generally* be found; there will *not always* be two triangles. For it may happen that CD is equal to the given quantity a; and then the two points marked B in the figure will both coincide with D, and there will be only *one* triangle with the given elements, namely the right-angled triangle ACD. Again, it may happen that CD is greater than the given quantity a, and then there is no triangle with the given elements.

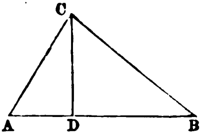
II. Suppose a equal to b.

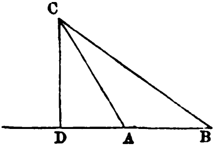


In this case the triangle is isosceles: we have $A = B$.

Then in the right-angled triangle ACD we know AC and the angle A; hence we can find AD. And AB is twice AD.

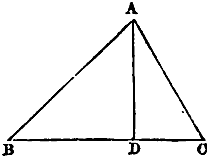
III. Suppose a greater than b.





Then as in I. we solve first the right-angled triangle ACD, and next the right-angled triangle BCD. The given angle A may be acute or obtuse; but there is only one triangle corresponding to the given elements. For if we were to take a point B' to the left of D, on BA produced, such that $DB' = DB$, we should have $CB' = CB = a$, but the angle CAB' of the triangle CAB' would not be equal to the given angle A: in the left-hand figure the angle A is acute while the angle CAB' would be obtuse, and in the right-hand figure the angle A is obtuse while the angle CAB' would be acute.

89. *To solve a triangle having given the three sides.*



Let a denote the side which is less than neither of the others, so that the angles B and C must be acute. Draw AD perpendicular to BC.

Let $BD = x$; then $DC = a - x$.

Now $AD^2 = AB^2 - BD^2 = AC^2 - DC^2$;

thus $c^2 - x^2 = b^2 - (a - x)^2 = b^2 - a^2 + 2ax - x^2$,

therefore $$x = \frac{c^2 + a^2 - b^2}{2a}.$$

Thus x is determined.

Then in the right-angled triangle ABD we know AB and BD; hence we can find the angle B. And in the right-angled triangle ACD we know AC and CD; hence we can find the angle C.

EXAMPLES. VIII.

Solve the following six triangles from the given quantities:

1. $c=84$, $B=45^\circ$, $C=30^\circ$.

2. $a=96$, $b=48$, $C=60^\circ$.

3. $a=1$, $b=1+\sqrt{3}$, $A=15^\circ$.

4. $a=\sqrt{2}$, $b=2(\sqrt{3}-1)$, $A=75^\circ$.

5. $a=\sqrt{3}$, $b=1$, $A=120^\circ$.

6. $a=10$, $b=5\sqrt{3}$, $c=5$.

In solving the following six triangles the Tables will be required:

7. $c=124{\cdot}5$, $A=56^\circ\,15'$, $B=48^\circ\,35'$.

8. $a=b=275{\cdot}6$, $C=72^\circ\,40'$.

9. $a=156$, $b=218$, $A=36^\circ$.

10. $a=750$, $b=625$, $A=80^\circ$.

11. $a=360$, $b=288$, $A=125^\circ$.

12. $a=125$, $b=170$, $c=200$.

13. In the *ambiguous case* when a, b, and A are given, shew that

$$c=b\cos A\pm\sqrt{(a^2-b^2\sin^2 A)}.$$

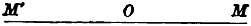
14. If the perpendicular drawn from the vertex of a triangle on the base fall within the triangle, shew that the difference of the segments of the base is to the difference of the sides as the sum of the sides is to the base.

Shew how to solve a triangle having given the two sides and the difference of the segments into which the base is divided by the perpendicular from the vertex.

IX. *Application of Algebraical Signs.*

90. In the preceding Chapters we have defined the Trigonometrical Ratios and established certain relations between them; and we have illustrated the use of the Trigonometrical Ratios. We have hitherto confined ourselves to angles not exceeding a right angle, but it is obvious that angles greater than a right angle may occur in mathematical investigations and in practice; and it becomes necessary to consider how the Trigonometrical Ratios apply to such angles.

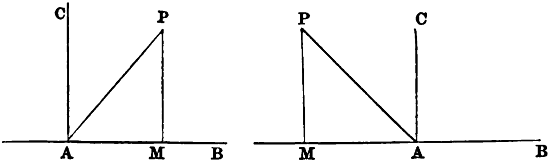
91. Let *O* be a fixed point in a fixed straight line, and suppose we have to determine the positions of other points in this straight line with respect to *O*. The position of any point in the straight line will be known if we know the distance of the point from *O*, and also know *on which side of O the point lies.* Now it is found convenient to adopt the following *convention;* distances measured in one direction from *O* along the fixed straight line are denoted by *positive* numbers, and distances measured in the opposite direction from *O* along the fixed straight line are denoted by *negative* numbers. Thus, for example, suppose that distances measured from *O* towards the *right hand* are denoted by positive numbers, and let *M* be a point the distance of which from *O* is denoted by 2 or $+2$; then if *M′* be as far from *O* as *M* is, and on the other side of *O*, the distance of *M′* from *O* is denoted by -2.



92. We have called this method of determining position by means of numbers affected with algebraical signs a *convention;* we mean by this word to indicate that it is not absolutely *necessary* to adopt this method, but merely *convenient.* The symbols + and − are defined in the beginning of elementary works on Algebra as indicative of the *operations* of addition and subtraction respectively.

As the student advances in Algebra he finds that the symbols + and − are also used as indicative of the *qualities* of quantities. And it is seen that no contradiction or confusion ultimately arises from this double mode of considering the symbols, but that Algebra gains thereby considerably in power. (See *Algebra,* Chaps. V. and XIV.)

93. We shall now extend our definitions of the Trigonometrical Ratios so as to make them applicable to any angle not greater than two right angles.



Let AB, AC be two straight lines at right angles; let a straight line turn round the point A from AB towards AC, and come into any position AP: draw PM perpendicular to AB or to AB produced through A. Then consider AP as always positive; consider AM as positive or negative according as M is on the same side of AC as B is, or on the opposite side; PM will in all cases be on the same side of AB as C is, and will be considered as positive. Let the angle PAB be denoted by A: then the Trigonometrical Ratios of A are thus defined,

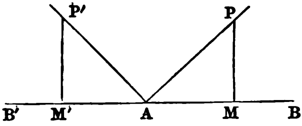
$$\sin A = \frac{PM}{AP}, \quad \tan A = \frac{PM}{AM}, \quad \sec A = \frac{AP}{AM},$$

$$\cos A = \frac{AM}{AP}, \quad \cot A = \frac{AM}{PM}, \quad \operatorname{cosec} A = \frac{AP}{PM},$$

$$\operatorname{vers} A = 1 - \cos A, \quad \operatorname{covers} A = 1 - \sin A.$$

94. The excess of two right angles over any angle is called the *supplement* of that angle. Thus if A be the number of degrees in any angle, $180-A$ is the number of degrees in the supplement of the angle.

95. *To compare the Trigonometrical Ratios of any angle with those of the supplement.*



Let PAB be any angle; produce BA to B', and make $P'AB'=PAB$: take $AP'=AP$ and draw PM and $P'M'$ perpendicular to BB'.

The angle $P'AB=180^\circ-P'AB'=180^\circ-PAB$; thus $P'AB$ is the supplement of PAB. The triangles PAM and $P'AM'$ are geometrically equal in all respects. Now, by definition,

$$\sin A=\frac{PM}{AP}, \quad \sin(180^\circ-A)=\frac{P'M'}{AP'};$$

and since PM and $P'M'$ are equal in magnitude and both positive, we have

$$\sin A=\sin(180^\circ-A).$$

Also, by definition,

$$\cos A=\frac{AM}{AP}, \quad \cos(180^\circ-A)=\frac{AM'}{AP'}.$$

Now AM and AM' are equal in magnitude, but since they are measured in opposite directions from A, they are of *opposite signs:*

thus $$\cos A=-\cos(180^\circ-A).$$

The other Trigonometrical Ratios of the angle A may be compared with those of the supplement either by direct use of the figure, or by employing the two results already established; thus, adopting the latter method,

$$\tan(180^\circ - A) = \frac{\sin(180^\circ - A)}{\cos(180^\circ - A)} = \frac{\sin A}{-\cos A} = -\tan A,$$

$$\cot(180^\circ - A) = \frac{\cos(180^\circ - A)}{\sin(180^\circ - A)} = \frac{-\cos A}{\sin A} = -\cot A,$$

$$\sec(180^\circ - A) = \frac{1}{\cos(180^\circ - A)} = \frac{1}{-\cos A} = -\sec A,$$

$$\operatorname{cosec}(180^\circ - A) = \frac{1}{\sin(180^\circ - A)} = \frac{1}{\sin A} = \operatorname{cosec} A,$$

$$\operatorname{vers}(180^\circ - A) = 1 - \cos(180^\circ - A) = 1 + \cos A.$$

Thus the sine and cosecant of any angle are respectively the same as the sine and cosecant of the supplement of the angle; the cosine, tangent, cotangent, and secant of any angle are *numerically* equal to the corresponding Ratios of the supplement of the angle, but are of opposite sign.

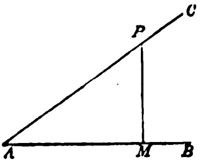
96. It follows from the preceding Article that if the sine of an angle be given, and we have to determine the angle without exceeding two right angles, there will be in general *two* angles which may be taken. For example, if the sine of an angle be $\frac{1}{2}$, we know by Art. 28 that 30° is one value of the corresponding angle; and by Art. 95 we have $180^\circ - 30^\circ$, that is 150°, for another value. Similar remarks apply for the cosecant. But for the other Trigonometrical Ratios this does not hold.

97. It is essential to have a distinct conception of the limits to which the values of the Trigonometrical Ratios tend, when the angle becomes very small.

The line AP is supposed to turn round A, remaining always of the same length.

First take the sine,

$$\sin PAB = \frac{PM}{AP}.$$



Now the figure shews that the smaller the angle *PAB* is the smaller *PM* is, that is the smaller is the sine; and by taking *PAB* small enough the sine will become as small as we please. These results are abbreviated thus:

the sine of 0^0 *is* 0.

Next take the cosine,

$$\cos PAB = \frac{AM}{AP}.$$

Now the figure shews that the smaller the angle *PAB* is the nearer is *AM* to *AP* in magnitude, that is, the nearer is the cosine to unity; and by taking *PAB* small enough the cosine will approach as near to unity as we please. These results are abbreviated thus:

the cosine of 0^0 *is* 1.

Next take the tangent,

$$\tan PAB = \frac{PM}{AM}.$$

Now the figure shews that the smaller the angle *PAB* is the smaller *PM* is and the nearer *AM* is to *AP*, that is, the smaller is the tangent; and by taking *PAB* small enough the tangent will become as small as we please. These results are abbreviated thus:

the tangent of 0^0 *is* 0.

Next take the cotangent,

$$\cot PAB = \frac{AM}{PM}.$$

Now as we have already stated, the smaller the angle PAB is the smaller PM is and the nearer AM is to AP, that is, the larger the cotangent is; and by taking PAB small enough the cotangent will become as great as we please. These results are abbreviated thus:

the cotangent of 0^0 *is* ∞.

Next take the cosecant,

$$\operatorname{cosec} PAB = \frac{AP}{PM}.$$

The reasoning is similar to that in the case of the cotangent, and the results may be abbreviated thus:

the cosecant of 0^0 *is* ∞.

Next take the secant,

$$\sec PAB = \frac{AP}{AM}.$$

The reasoning is similar to that in the case of the cosine, and the results may be abbreviated thus:

the secant of 0^0 *is* 1.

Since vers $PAB = 1 - \cos PAB$ we have results which we may abbreviate thus:

the versed sine of 0^0 *is* 0.

98. We have obtained the previous results by repeated reference to the figure, but it should be observed that from the first two results the others might have been inferred. Thus, for example,

since $$\tan PAB = \frac{\sin PAB}{\cos PAB},$$

the tangent of 0^0 may be said to be equal to $\frac{0}{1}$, that is to 0.

Again $$\cot PAB = \frac{\cos PAB}{\sin PAB};$$

thus the cotangent of 0^0 may be said to be equal to $\frac{1}{0}$, that is to infinity.

99. It is also necessary to have a distinct conception of the limits to which the Trigonometrical Ratios tend when the angle becomes very nearly a right angle. These may be obtained from the figure, in the manner of Art. 97; or they may be deduced from the results given in Art. 97. We shall adopt the latter method.

Thus, by Art. 16,

$$\begin{aligned}
\sin 90^0 &= \cos 0^0 = 1, \\
\cos 90^0 &= \sin 0^0 = 0, \\
\tan 90^0 &= \cot 0^0 = \infty, \\
\cot 90^0 &= \tan 0^0 = 0, \\
\sec 90^0 &= \operatorname{cosec} 0^0 = \infty, \\
\operatorname{cosec} 90^0 &= \sec 0^0 = 1, \\
\operatorname{vers} 90^0 &= 1 - \cos 90^0 = 1.
\end{aligned}$$

100. Finally it is necessary to have a distinct conception of the limits to which the Trigonometrical Ratios tend when the angle becomes very nearly two right angles. These also may be obtained from the figure in the manner of Art. 97; or they may be deduced from the results given in Art. 97. We shall adopt the latter method.

Thus, by Art. 95,

$$\begin{aligned}
\sin 180^0 &= \sin 0^0 = 0, \\
\cos 180^0 &= -\cos 0^0 = -1, \\
\tan 180^0 &= -\tan 0^0 = 0, \\
\cot 180^0 &= -\cot 0^0 = \infty, \\
\sec 180^0 &= -\sec 0^0 = -1, \\
\operatorname{cosec} 180^0 &= \operatorname{cosec} 0^0 = \infty, \\
\operatorname{vers} 180^0 &= 1 - \cos 180^0 = 2.
\end{aligned}$$

101. One remark may be made to prevent a possible misconception of some of the preceding results. We have put ∞ as the value of tan 90°. The student must not assume that ∞ means necessarily $+\infty$. If an angle is a little less than 90°, the tangent is large and *positive;* if the angle is a little greater than 90°, the tangent is large and *negative;* so that in saying that tan 90° is ∞ we must not suppose that the sign $+$ is necessarily to be taken before ∞. So also in other cases. We have put ∞ as the value of cot 180°, whereas the student might have expected $-\infty$. It is true that if an angle is a little less than 180°, the cotangent is negative; but if, as in a subsequent Chapter, we suppose an angle a little greater than 180°, the cotangent is positive.

102. The following table collects many of the results obtained in the present Chapter and Chapter III.

	0°	30°	45°	60°	90°	120°	135°	150°	180°
sine	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0
cot	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	-1	$-\sqrt{3}$	∞
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞	-2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$	-1
cosec	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞

EXAMPLES. IX.

1. Find the number of degrees in each angle of a regular pentagon. Find also the number of grades.

2. The number of grades in an angle is equal to two-thirds of the number of degrees in the supplement of the angle: determine the angle.

3. There are as many degrees in the complement of A as in the supplement of $4A$: determine A.

4. Given $\sin A = \frac{5}{13}$, find the Trigonometrical Ratios for $\frac{1}{2}A$.

5. Given $\cot A = 2 - \sqrt{3}$, shew that $\sec A = \sqrt{6} + \sqrt{2}$, and $\operatorname{cosec} A = \sqrt{6} - \sqrt{2}$.

6. If $\tan A = 1 + \sqrt{2}$, find $\cos^2 A$, $\sin^2 A$, and $\cos 2A$.

7. If $\tan A \tan B = 1$, shew that $\sec A = \operatorname{cosec} B$.

8. If $\tan 2A = -\frac{3}{4}$, find $\cos 2A$, $\sin 2A$, $\cos A$, and $\sin A$.

9. Shew that $\tan 52\frac{1}{2}^\circ = (\sqrt{3} + \sqrt{2})(\sqrt{2} - 1)$.

10. Given $\sin A = m \sin B$, and $\cos A = n \cos B$, find $\sin^2 A$ and $\sin^2 B$.

11. Given $\sin A = m \sin B$, and $\tan A = n \tan B$, find $\sin^2 A$ and $\cos^2 B$.

12. If $\tan A + \sec A = \frac{3}{2}$, shew that $\sin A = \frac{5}{13}$.

13. If $\tan A + \sec A = a$, shew that $\sin A = \frac{a^2 - 1}{a^2 + 1}$.

14. Given $\sin A + \operatorname{cosec} A = 2\frac{1}{1640}$, find $\sin A$.

15. If $\sin A + \operatorname{cosec} A = a$, shew that

$$\sin A = \frac{a - \sqrt{(a^2 - 4)}}{2}.$$

16. If $$a \cos^2 x + b \sin^2 x = m \cos^2 y,$$
and $$a \sin^2 x + b \cos^2 x = n \sin^2 y,$$
find $\sin^2 x$ and $\sin^2 y$.

17. Shew that

$$\frac{\tan A}{\tan A - \tan B} + \frac{\cot A}{\cot A - \cot B} = 1.$$

18. Shew that

$$(4 \cos^2 A - 1)^2 \tan^2 A + (3 - 4 \cos^2 A)^2 = \sec^2 A.$$

19. Shew geometrically that $\sin 2A$ is less than $2 \sin A$.

20. Shew that

$$\operatorname{cosec} A (\sec A - 1) + \sin A = \cot A (1 - \cos A) + \tan A.$$

21. Given $a = 5$, $b = 20$, $C = 90^\circ$, find A and B.

$\log 5 = \cdot 6989700$, $L \tan 75^\circ 57' = 10 \cdot 6016170$,
$L \tan 75^\circ 58' = 10 \cdot 6021537$.

22. Given $a = 9 \cdot 65$, $b = 12 \cdot 24$, $C = 90^\circ$, find A and B.

$\log 2 = \cdot 3010300$, $\log 153 = 2 \cdot 1846914$, $\log 193 = 2 \cdot 2855573$,

$L \tan 38^\circ 15' = 9 \cdot 8967116$, $L \tan 38^\circ 16' = 9 \cdot 8969714$.

23. If $\sin A + \cos A = a + \sqrt{(1 - a^2)}$, shew that either $\sin A$ or $\cos A$ is equal to a.

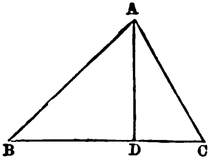
24. If $\sin A = a + \sqrt{\left(\frac{1}{2} - a^2\right)}$, find $\cos A$.

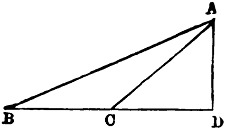
X. *Properties of Triangles.*

103. The present Chapter will contain some properties of triangles which are useful for the solution of triangles. We begin with a proposition which we have already given in Art. 37, but which must be repeated in order to shew that it holds for oblique-angled triangles as well as for acute-angled triangles.

We retain the notation of Art. 37.

104. *In any triangle the sides are proportional to the sines of the opposite angles.*





Let ABC be a triangle, and from A draw AD perpendicular to the opposite side, meeting that side, or that side produced at D.

If B and C are *acute* angles, we have from the left-hand figure

$$AD = AB \sin B, \text{ and } AD = AC \sin C;$$

therefore $$AB \sin B = AC \sin C,$$

therefore $$\frac{AB}{AC} = \frac{\sin C}{\sin B},$$

that is $$\frac{c}{b} = \frac{\sin C}{\sin B}.$$

If the angle C be *obtuse*, we have from the right-hand figure

$$AD = AB \sin B, \text{ and } AD = AC \sin(180^\circ - C)$$
$$= AC \sin C \text{ (Art. 95)};$$

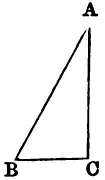
therefore $$AB \sin B = AC \sin C,$$

therefore $$\frac{AB}{AC} = \frac{\sin C}{\sin B},$$

that is $$\frac{c}{b} = \frac{\sin C}{\sin B}.$$

If the angle C be a *right* angle we have

$$AC = AB \sin B,$$



therefore $$\frac{AB}{AC} = \frac{1}{\sin B} = \frac{\sin 90^\circ}{\sin B} \quad \text{(Art. 99)},$$

that is $$\frac{c}{b} = \frac{\sin C}{\sin B}.$$

Thus it is shewn that in every case

$$\frac{c}{b} = \frac{\sin C}{\sin B}.$$

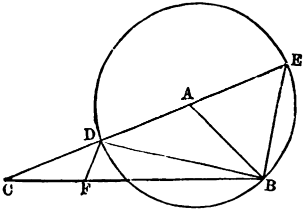
Similarly $$\frac{a}{b} = \frac{\sin A}{\sin B}, \text{ and } \frac{a}{c} = \frac{\sin A}{\sin C}.$$

The results may be written symmetrically thus:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c};$$

and we shall shew hereafter that each of these is equal to $\frac{1}{2R}$, where R is the radius of the circle described round the triangle.

105. *In any triangle* ABC *the tangent of half the difference of the angles* B *and* C *is to the tangent of half their sum as the difference of the two sides* AB *and* AC *is to their sum.*



Let AB be the shorter of the two sides AB, AC. With centre A, and radius AB, describe a circle cutting AC at D. Produce CA to meet the circumference again at E. Draw DF at right angles to BD, meeting BC at F.

The angle BAE is equal to the sum of the angles ABC and ACB, by Euclid I. 32; the angle BDE is equal to half the angle BAE, by Euclid III. 20;

thus $$BDE = \frac{1}{2}(B + C).$$

The angle ADB is equal to the sum of the angles DBC and DCB, by Euclid I. 32:

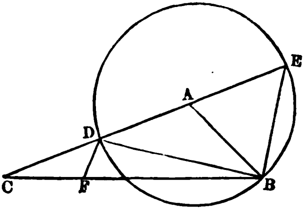
thus $$DBC = ADB - DCB$$
$$= \frac{1}{2}(B + C) - C = \frac{1}{2}(B - C).$$

The angle DBE is a right angle, by Euclid III. 31:

thus $$\frac{BE}{BD} = \tan BDE.$$

And $$\frac{DF}{DB} = \tan DBC.$$

Therefore $$\frac{\tan\frac{1}{2}(B - C)}{\tan\frac{1}{2}(B + C)} = \frac{DF}{DB} \div \frac{BE}{DB} = \frac{DF}{BE}.$$



But DF is parallel to BE, by Euclid I. 29; therefore, by Euclid VI. 4,

$$\frac{DF}{DC}=\frac{BE}{CE},$$

therefore $$\frac{DF}{BE}=\frac{CD}{CE}=\frac{CA-AD}{CA+AE}=\frac{CA-AB}{CA+AB}:$$

thus $$\frac{\tan\frac{1}{2}(B-C)}{\tan\frac{1}{2}(B+C)}=\frac{b-c}{b+c}.$$

The above investigation has been usually given in elementary works on Trigonometry; another method has been indicated in which two important formulæ are derived from the figure, and then the result just obtained is deduced.

We have shewn that the angle $ADB=\frac{1}{2}(B+C)$, and that the angle $DBC=\frac{1}{2}(B-C)$; thus $CBE=90^\circ+\frac{1}{2}(B-C)$. And the angle $CEB=\frac{1}{2}A$, by Euclid III. 20.

Now from the triangle ECB we have

$$\frac{CE}{CB}=\frac{\sin CBE}{\sin CEB}=\frac{\sin(180^\circ-CBE)}{\sin\frac{1}{2}A}=\frac{\sin\{90^\circ-\frac{1}{2}(B-C)\}}{\sin\frac{1}{2}A};$$

that is $$\frac{b+c}{a}=\frac{\cos\frac{1}{2}(B-C)}{\sin\frac{1}{2}A};$$

therefore $$(b+c)\sin\frac{1}{2}A=a\cos\frac{1}{2}(B-C)\dots\dots\dots\dots\dots(1).$$

Again, from the triangle DCB we have

$$\frac{CD}{CB} = \frac{\sin CBD}{\sin CDB} = \frac{\sin \frac{1}{2}(B-C)}{\sin \frac{1}{2}(B+C)},$$

that is $$\frac{b-c}{a} = \frac{\sin \frac{1}{2}(B-C)}{\cos \frac{1}{2}A} \quad \text{(Art. 16)};$$

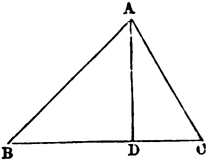
therefore $$(b-c)\cos \tfrac{1}{2}A = a \sin \tfrac{1}{2}(B-C) \ldots\ldots\ldots\ldots\ldots(2).$$

Divide (2) by (1); then

$$\tan \tfrac{1}{2}(B-C) = \frac{b-c}{b+c}\cot \tfrac{1}{2}A = \frac{b-c}{b+c}\tan \tfrac{1}{2}(B+C) \quad \text{(Art. 16)};$$

this agrees with the result already obtained.

106. *To express the cosine of an angle of a triangle in terms of the sides.*



Let ABC be a triangle, and suppose C an acute angle. Draw AD perpendicular to BC.

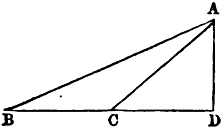
Then, by Euclid II. 13,

$$AB^2 = BC^2 + AC^2 - 2BC \,.\, CD,$$

and $$CD = AC \cos C;$$

therefore $$c^2 = a^2 + b^2 - 2ab \cos C.$$

Next suppose C an obtuse angle. Draw AD perpendicular to BC produced.



Then, by Euclid II. 12,

$$AB^2 = BC^2 + AC^2 + 2BC \,.\, CD;$$

and $CD = AC \cos ACD = AC \cos (180^0 - C) = -AC \cos C$, by Art. 95;

therefore $$c^2 = a^2 + b^2 - 2ab \cos C.$$

Thus in both cases

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

Moreover when C is a right angle $a^2 + b^2 = c^2$, and $\cos C = 0$, by Art. 99. Thus the formula just found for $\cos C$ is true whatever the angle C may be.

Similarly

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \quad \cos B = \frac{c^2 + a^2 - b^2}{2ac}.$$

107. *To express the sine, cosine, and tangent, of half an angle of a triangle in terms of the sides.*

We have by the preceding Article

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc};$$

therefore $$1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc} = \frac{a^2 - (b-c)^2}{2bc}.$$

Hence, by Art. 35,

$$\left(\sin \tfrac{1}{2} A\right)^2 = \frac{a^2-(b-c)^2}{4bc}$$

$$= \frac{(a+b-c)(a+c-b)}{4bc}.$$

Let $2s$ stand for $a+b+c$, so that s is half the sum of the sides of the triangle; then

$$a+b-c=a+b+c-2c=2s-2c=2(s-c),$$

$$a+c-b=a+b+c-2b=2s-2b=2(s-b).$$

Therefore $$\left(\sin \tfrac{1}{2} A\right)^2 = \frac{(s-b)(s-c)}{bc},$$

and $$\sin \tfrac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{bc}}.$$

Also $$1+\cos A = 1+\frac{b^2+c^2-a^2}{2bc} = \frac{(b+c)^2-a^2}{2bc}.$$

Hence, by Art. 35,

$$\left(\cos \tfrac{1}{2} A\right)^2 = \frac{(b+c)^2-a^2}{4bc}$$

$$= \frac{(a+b+c)(b+c-a)}{4bc} = \frac{s(s-a)}{bc};$$

and $$\cos \tfrac{1}{2} A = \sqrt{\frac{s(s-a)}{bc}}.$$

From the values of $\sin \tfrac{1}{2} A$ and $\cos \tfrac{1}{2} A$ we deduce

$$\tan \tfrac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

108. *To express the sine of the angle of a triangle in terms of the sides.*

By Art. 35, $\sin A = 2\sin\frac{1}{2}A\cos\frac{1}{2}A$,

$$\text{therefore } \sin A = 2\sqrt{\frac{(s-b)(s-c)}{bc}} \times \sqrt{\frac{s(s-a)}{bc}}$$

$$= \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.$$

Or we may proceed thus:

$$(\sin A)^2 = 1 - (\cos A)^2 = 1 - \left(\frac{b^2+c^2-a^2}{2bc}\right)^2$$

$$= \frac{4b^2c^2 - (b^2+c^2)^2 - a^4 + 2a^2(b^2+c^2)}{4b^2c^2}$$

$$= \frac{2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4}{4b^2c^2};$$

therefore

$$\sin A = \frac{\sqrt{2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4}}{2bc}.$$

By comparing the two expressions for sin A we infer that

$$s(s-a)(s-b)(s-c) = \frac{2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4}{16},$$

and this can be verified by multiplying out the factors s, $s-a$, $s-b$, and $s-c$.

EXAMPLES. X.

1. If $\sin B = \frac{1}{4}$, $a = 3$, $b = \frac{3}{2}$, find A.

2. If $A = 75^\circ$, $B = 45^\circ$, $b = 2$, shew that $a = \sqrt{3} + 1$.

3. If $b = c\,(\sqrt{3} - 1)$, and $A = 30^\circ$, find B and C.

4. If $b = 2a$, and $C = 60^\circ$, find A, B, and c.

5. Find A when $a = 7$, $b = 5$, $c = 3$.

6. If a, b, and c are $1\frac{1}{5}$ feet, $1\frac{3}{5}$ feet, and 2 feet respectively, find C.

7. The sides of a triangle are 7, 8, 13: find the greatest angle.

8. The sides of a triangle are respectively 13 and 15 feet, and the cosine of the included angle is $\frac{33}{65}$: find the third side, and also the perpendicular on it from the given angle.

9. Shew from the formulæ for $\sin B$ and $\sin \frac{C}{2}$ that

$B = \frac{C}{2}$ if $c^2 = b\,(b + a)$.

10. If $a = 5$, $b = 4$, and $C = 60^\circ$, find c; having given

$\log 3 = \cdot 4771213$, $\log 7 = \cdot 8450980$.

$\log 45825 = 4\cdot 6611025$, $\log 45826 = 4\cdot 6611120$.

11. A perpendicular is drawn from the angle A of a triangle on the side BC meeting it at D; and a perpendicular from B on the side CA meeting it at E: shew that $DE = c \cos C$.

12. Shew immediately from the figure in Art. 106 that

$$a = b \cos C + c \cos B.$$

13. From the result in the preceding example and the two analogous results deduce the value of $\cos C$ given in Art. 106.

14. If $\sin A = p\sqrt{(1-q^2)} + q\sqrt{(1-p^2)}$, find $\cos A$.

15. Shew that $\tan A = \dfrac{\sin A + \sin 2A}{1 + \cos A + \cos 2A}$.

16. If $\tan A + \cot A = 2$, then $\sin A + \cos A = \sqrt{2}$.

17. Find A from the equation

$$\sec^2 A + \operatorname{cosec}^2 A = 3 \sec^4 A.$$

18. The hypotenuse AB of a right-angled triangle is divided at D so that AD is to BD as CB is to CA : shew

that $$\tan ACD = \frac{a^2}{b^2}, \qquad CD = \frac{\sqrt{a^4+b^4}}{a+b}.$$

19. From a ship at sea it is observed that the angle between two forts A and B is α; the ship sails for m miles towards A, and the angle between the forts is then observed to be β; find the distance of the ship from B at the second observation.

20. If $$a \cos^2 x + b \sin^2 x = m \cos^2 y,$$

$$a \sin^2 x + b \cos^2 x = n \sin^2 y,$$

and $$m \tan^2 x = n \tan^2 y,$$

shew that $$(a+b)(m+n) = 2mn,$$

and $$\tan^2 x = 1.$$

XI. *Solution of Triangles.*

109. *To solve a triangle having given two angles and a side.*

Suppose A and C the given angles, and b the given side; then

$$B = 180^0 - A - C;$$

$$\frac{a}{b} = \frac{\sin A}{\sin B}, \qquad \text{(Art. 104)}$$

therefore $$a = \frac{b \sin A}{\sin B};$$

therefore $$\log a = \log b + \log \sin A - \log \sin B$$
$$= \log b + L \sin A - 10 - L \sin B + 10$$
$$= \log b + L \sin A - L \sin B.$$

Similarly $\log c = \log b + L \sin C - L \sin B.$

110. *To solve a triangle having given two sides and the included angle.*

Suppose b and c the given sides and A the included angle.

We have, by Art. 105,

$$\frac{\tan \frac{1}{2}(B-C)}{\tan \frac{1}{2}(B+C)} = \frac{b-c}{b+c}.$$

Now $$A + B + C = 180^0,$$

therefore $$B + C = 180^0 - A,$$

therefore $$\tfrac{1}{2}(B+C)=90^\circ-\frac{A}{2},$$

therefore $$\tan\tfrac{1}{2}(B+C)=\cot\frac{A}{2}. \quad \text{(Art. 16.)}$$

Hence $$\frac{\tan\frac{1}{2}(B-C)}{\cot\frac{1}{2}A}=\frac{b-c}{b+c};$$

therefore $$\tan\tfrac{1}{2}(B-C)=\frac{b-c}{b+c}\cot\tfrac{1}{2}A,$$

therefore

$$\log\tan\tfrac{1}{2}(B-C)=\log(b-c)+\log\cot\tfrac{1}{2}A-\log(b+c),$$

therefore

$$L\tan\tfrac{1}{2}(B-C)=\log(b-c)+L\cot\tfrac{1}{2}A-\log(b+c).$$

This formula determines $\frac{1}{2}(B-C)$; and $\frac{1}{2}(B+C)$ is known, since it is $90^\circ-\frac{1}{2}A$; then B and C can be immediately found.

Also $$\frac{a}{c}=\frac{\sin A}{\sin C},$$

therefore $$\log a=\log c+L\sin A-L\sin C,$$

from which a can be found.

Or a may be found from one of the formulæ investigated in the latter part of Art. 105. We have

$$a\cos\tfrac{1}{2}(B-C)=(b+c)\sin\tfrac{1}{2}A,$$

therefore $\log a=\log(b+c)+L\sin\frac{1}{2}A-L\cos\frac{1}{2}(B-C)$.

In this way of finding a we only require *two* additional logarithms, as $\log(b+c)$ is already known; whereas in the former way we required *three* additional logarithms: moreover $L\sin\frac{1}{2}A$ can be taken out of the tables at the same opening of them as $L\cot\frac{1}{2}A$, thus saving trouble.

These processes suppose b and c to be unequal. If b and c are equal, the triangle is isosceles; and then by drawing a perpendicular from the angle A to the base the triangle may be divided into two right-angled triangles, and the solution may be completed as in Art. 78.

111. *To solve a triangle having given two sides and the angle opposite to one of them.*

Let a and b be the givon sides, and A the given angle; then

$$\frac{\sin B}{\sin A} = \frac{b}{a},$$

therefore $$\sin B = \frac{b}{a} \sin A,$$

therefore $$\log \sin B = \log b + \log \sin A - \log a,$$

therefore $$L \sin B = \log b + L \sin A - \log a.$$

If $\frac{b \sin A}{a}$ is less than unity, two different angles may be found less than 180^0 which have $\frac{b \sin A}{a}$ for sine, one of these angles being less than a right angle, and the other greater. If a be not less than b, then A must be not less than B, and therefore B must be an acute angle; thus only the smaller value is admissible for B. If a be less than b, then either value may be taken for B. When B is determined, C is known, since it is $180^0 - A - B$, and then c can be found from

$$\frac{c}{a} = \frac{\sin C}{\sin A}.$$

Thus if two values are admissible for B we obtain two corresponding values for C and c, so that two triangles can be found from the given elements.

If $\frac{b \sin A}{a} = 1$, then B is a right angle, so that only one triangle can be found from the given elements.

If $\frac{b \sin A}{a}$ is greater than unity, no triangle exists with the given elements.

Thus when two sides are given and the angle opposite to the less, we can *generally* find two triangles from the given elements, and this case in the solution of triangles is therefore called the *ambiguous* case.

We say that two triangles can be *generally* found, in order to allow for the exceptions: for the triangle may be *right-angled*, and then only one triangle can be found; or the triangle may be impossible.

Some figures illustrating the solution of triangles when the given elements are two sides and an opposite angle will be found in Art. 88.

112. *To solve a triangle having given the three sides.*

By Art. 107, we have

$$\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}},$$

$$\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}},$$

$$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}},$$

and similar formulæ are true for the other half angles. Any one of the three formulæ will serve for finding the angle $\frac{A}{2}$; the formulæ for the tangent will however be the best to use with logarithms, because then we only require the logarithms of s, $s-a$, $s-b$, and $s-c$ in order to find *all* the angles; whereas if we use the formulæ for the sine or cosine, we shall require also the logarithms of the sides.

113. We will now take some examples of the solution of triangles; we shall not give the *process* of applying the principle of proportional parts, for this has been sufficiently exemplified in Chapters VI. and VII.: we shall merely state the results.

Ex. (1). Given $b=234{\cdot}7$, $c=185{\cdot}4$, $A=84^0\ 36'$,

$$L\tan\tfrac{1}{2}(B-C)=\log(b-c)+L\cot\tfrac{1}{2}A-\log(b+c).$$

$$b-c=49{\cdot}3,\qquad b+c=420{\cdot}1,\qquad \tfrac{1}{2}A=42^0\ 18'.$$

$$\begin{aligned}\text{Log } 49{\cdot}3 &= \ \ 1{\cdot}6928469\\ L\cot 42^0 18' &= 10{\cdot}0409920\\ \hline &\ \ \ 11{\cdot}7338389\\ \log 420{\cdot}1 &= \ \ 2{\cdot}6233527\\ \hline L\tan\tfrac{1}{2}(B-C) &= \ \ 9{\cdot}1104862\end{aligned}$$

Hence we find from the tables

$$\tfrac{1}{2}(B-C)=7^0\ 20'\ 56''\text{ nearly,}$$

and $$\tfrac{1}{2}(B+C)=47^0\ 42'.$$

Thus $B=55^0\ 2'\ 56''$, $C=40^0\ 21'\ 4''$.

Then $$\log a=\log c+L\sin A-L\sin C$$

$$\begin{aligned}\log 185{\cdot}4 &= 2{\cdot}2681097\\ L\sin 84^0\ 36' &= 9{\cdot}9980683\\ \hline &\ \ 12{\cdot}2661780\\ L\sin 40^0\ 21'\ 4'' &= 9{\cdot}8112196\\ \hline \log a &= 2{\cdot}4549584\end{aligned}$$

Hence we find $a=285{\cdot}0745$.

Or we may find a by the other method explained in Art. 110.

$$\log a = \log(b+c) + L\sin\tfrac{1}{2}A - L\cos\tfrac{1}{2}(B-C)$$

$$\begin{array}{rl} \log 420\cdot1 = & 2\cdot6233527 \\ L\sin 42^\circ\ 18' = & \underline{9\cdot8280231} \\ & 12\cdot4513758 \\ L\cos 7^\circ\ 20'\ 56'' = & \underline{9\cdot9964178} \\ \log a = & 2\cdot4549580 \end{array}$$

Hence we can find a as before.

The two modes of calculating log a give results which differ slightly; but $\tfrac{1}{2}(B-C)$ is in fact rather less than $7^\circ\ 20'\ 56''$, being about $7^\circ\ 20'\ 55''\cdot8$: if this more correct value be employed we shall find that both modes of calculating will agree in giving a result almost identical with that just obtained.

Ex. (2). Given $a = 283\cdot4$, $b = 348\cdot5$, $A = 32^\circ\ 15'$.

$$L\sin B = \log b + L\sin A - \log a,$$

$$\begin{array}{rl} \log 348\cdot5 = & 2\cdot5422028 \\ L\sin 32^\circ\ 15' = & \underline{9\cdot7272276} \\ & 12\cdot2694304 \\ \log 283\cdot4 = & \underline{2\cdot4523998} \\ L\sin B = & 9\cdot8170306 \end{array}$$

$$B = 41^\circ\ 0'\ 36'', \text{ or } 138^\circ\ 59'\ 24''.$$

This is an example of the *ambiguous case*, so that there are two triangles with the given elements.

If we take $B = 41^\circ\ 0'\ 36''$ we have

$$C = 106^\circ\ 44'\ 24''.$$

Then $\log c = \log a + L\sin C - L\sin A$,

$$\begin{array}{rl} \log 283\cdot4 = & 2\cdot4523998 \\ L\sin 106^\circ\ 44'\ 24'' = L\sin 73^\circ\ 15'\ 36'' = & \underline{9\cdot9811940} \\ & 12\cdot4335938 \\ L\sin 32^\circ\ 15' = & \underline{9\cdot7272276} \\ \log c = & 2\cdot7063662 \end{array}$$

Hence we find $c = 508{\cdot}588$.

If we take $B = 138^0\,59'\,24''$, we have

$$C = \quad 8^0\,45'\,36''.$$

$$\begin{array}{rr} \log 283{\cdot}4 = & 2{\cdot}4523998 \\ L\sin 8^0\,45'\,36'' = & \underline{9{\cdot}1826882} \\ & 11{\cdot}6350880 \\ L\sin 32^0\,15' = & \underline{9{\cdot}7272276} \\ \log c = & 1{\cdot}9078604 \end{array}$$

Hence we find $c = 80{\cdot}8836$.

Ex. (3). Given $a = 15$, $b = 16$, $c = 17$.

Here $$s = \frac{1}{2}(15 + 16 + 17) = 24,$$

$$s - a = 9, \quad s - b = 8, \quad s - c = 7.$$

$$\begin{array}{rr} \log 8 = & {\cdot}9030900 \\ \log 7 = & \underline{{\cdot}8450980} \\ & 1{\cdot}7481880 \end{array} \qquad \begin{array}{rr} \log 24 = & 1{\cdot}3802112 \\ \log\ \ 9 = & \underline{{\cdot}9542425} \\ & 2{\cdot}3344537 \\ & \underline{1{\cdot}7481880} \\ & 2)\underline{{\cdot}5862657} \\ & {\cdot}2931328 \end{array}$$

$$L\tan\frac{A}{2} = 10 + \frac{1{\cdot}7481880}{2} - \frac{2{\cdot}3344537}{2}$$

$$= 10 - {\cdot}2931328 = 9{\cdot}7068672.$$

Hence we find $\dfrac{A}{2} = 26^0\,59'\,3''$.

Similarly we find $\dfrac{B}{2} = 29^0\,48'\,18''$.

Hence $$\frac{C}{2} = 90^0 - \frac{A}{2} - \frac{B}{2} = 33^0\,12'\,39''.$$

Examples. XI.

1. Given $a=254$, $B=16^\circ$, $C=64^\circ$: find b.

$\log 2{\cdot}54={\cdot}4048337$, $L\sin 80^\circ=9{\cdot}9933515$,
$\log 7{\cdot}109={\cdot}8518085$, $L\sin 16^\circ=9{\cdot}4403381$.
$\log 7{\cdot}11={\cdot}8518696$,

2. Given $a=274$, $A=78^\circ$, $B=54^\circ$: find b.

$\log 274=2{\cdot}4377506$, $L\sin 78^\circ=9{\cdot}9904044$,
$\log 226{\cdot}62=2{\cdot}3552982$, $L\sin 54^\circ=9{\cdot}9079576$.
$\log 226{\cdot}63=2{\cdot}3553174$,

3. Given $a=1000$, $B=104^\circ$, $C=24^\circ\, 29'\, 20''$: find b.

$\log 12396=4{\cdot}0932816$, $L\sin 76^\circ=9{\cdot}9869041$,
$\log 12397=4{\cdot}0933166$, $L\sin 51^\circ\, 30'=9{\cdot}8935444$,
$L\sin 51^\circ\, 31'=9{\cdot}8936448$.

4. Given $b=55$, $c=45$, $A=6^\circ$: find B and C.

$L\tan 87^\circ=11{\cdot}2806042$, $L\tan 62^\circ\, 20'=10{\cdot}2804451$,
$L\tan 62^\circ\, 21'=10{\cdot}2807524$.

5. Given $b=21$, $c=10{\cdot}5$, $A=36^\circ\, 52'\, 12''$: find B and C.

$\log 2={\cdot}3010300$, $L\cot 18^\circ\, 26'\, 6''=10{\cdot}4771213$.
$\log 15=1{\cdot}1760913$,

6. Given $b=426$, $c=354$, $A=49^\circ\, 16'$: find B and C.

$\log 78=1{\cdot}8920946$, $L\cot 24^\circ\, 38'=10{\cdot}3386231$,
$\log 72=1{\cdot}8573325$, $L\tan 11^\circ\, 22'=9{\cdot}3032609$,
$L\tan 11^\circ\, 23'=9{\cdot}3039143$.

7. Given $b=100$, $c=60$, $A=42^\circ\, 30'$: find B and C.

$\log 2={\cdot}3010300$, $L\tan 32^\circ\, 44'=9{\cdot}8080829$,
$L\cot 21^\circ\, 15'=10{\cdot}4101858$, $L\tan 32^\circ\, 45'=9{\cdot}8083606$.

8. Given $b=82471$, $c=63529$, $A=43^\circ\ 10'$: find B and C.

$\log 18942 = 4{\cdot}2774258$, $L \tan 68^\circ\ 25' = 10{\cdot}4027530$,
$\log 146 = 2{\cdot}1643529$, $L \tan 18^\circ\ 9'\ 20'' = 9{\cdot}5157731$,
$L \tan 18^\circ\ 9'\ 30'' = 9{\cdot}5158442$.

9. Given $b=375400{\cdot}1$, $c=327762{\cdot}9$, $A=57^\circ\ 53'\ 16''{\cdot}8$: find B and C.

$\log 703163 = 5{\cdot}8470561$, $L \cot \frac{A}{2} = 10{\cdot}2572497$,
$\log 47637{\cdot}2 = 4{\cdot}6779462$, $L \tan 6^\circ\ 59'\ 2''{\cdot}4 = 9{\cdot}0881398$.

10. Given $b=5{\cdot}75$, $c=4{\cdot}5$, $A=48^\circ\ 20'$: find B, C, and a.

$\log 1{\cdot}25 = {\cdot}0969100$, $L \tan 65^\circ\ 50' = 10{\cdot}3480258$,
$\log 4{\cdot}3485 = {\cdot}6383382$, $L \tan 15^\circ\ 12'\ 16'' = 9{\cdot}4342119$,
$\log 4{\cdot}5 = {\cdot}6532125$, $L \sin 48^\circ\ 20' = 9{\cdot}8733352$,
$\log 10{\cdot}25 = 1{\cdot}0107239$, $L \sin 50^\circ\ 37'\ 44'' = 9{\cdot}8882095$.

11. Given $a=528$, $b=252$, $A=124^\circ\ 34'$: find B and C.

$\log 252 = 2{\cdot}4014005$, $L \sin 55^\circ\ 26' = 9{\cdot}9156460$,
$\log 528 = 2{\cdot}7226339$, $L \sin 23^\circ\ 8' = 9{\cdot}5942513$,
$L \sin 23^\circ\ 9' = 9{\cdot}5945469$.

12. Given $a=120$, $b=80$, $A=60^\circ$: find B, C and c.
$\log 2 = {\cdot}3010300$, $L \sin 35^\circ\ 15' = 9{\cdot}7612851$,
$\log 3 = {\cdot}4771213$, $L \sin 35^\circ\ 16' = 9{\cdot}7614638$,
$\log 1{\cdot}3797 = {\cdot}1397847$, $L \sin 84^\circ\ 44' = 9{\cdot}9981626$,
$\log 1{\cdot}3798 = {\cdot}1398161$, $L \sin 84^\circ\ 45' = 9{\cdot}9981743$.

13. Given $a=21{\cdot}217$, $b=12{\cdot}543$, $A=29^\circ\ 51'$: find B and C.

$\log 21217 = 4{\cdot}3266840$, $L \sin 17^\circ\ 6'\ 40'' = 9{\cdot}4686806$,
$\log 12543 = 4{\cdot}0984014$, $L \sin 17^\circ\ 6'\ 50'' = 9{\cdot}4687490$,
$L \sin 29^\circ\ 51' = 9{\cdot}6969947$.

14. Given $\log a = {\cdot}9717973$, $\log b = {\cdot}8929345$,
$A = 70^\circ\ 12'\ 10''$: find B, C, and $\log c$.

$L \sin 70^\circ\ 12'\ 10'' = 9{\cdot}9735422$, $L \sin 58^\circ\ 6'\ 30'' = 9{\cdot}9289325$.
$L \sin 51^\circ\ 41'\ 20'' = 9{\cdot}8946794$,

15. Given $a = 36$, $b = 44$, $A = 32^\circ\ 42'$: find B and C.

$\log 3 = {\cdot}4771213$, $L \sin 32^\circ\ 42' = 9{\cdot}7325870$,
$\log 11 = 1{\cdot}0413927$, $L \sin 41^\circ\ 19' = 9{\cdot}8196888$,
$L \sin 41^\circ\ 20' = 9{\cdot}8198325$.

16. Given $a = 10$, $b = 15$, $L \sin A = 9{\cdot}5228787$,
$\log 3 = {\cdot}4771213$: find B.

17. Given $a = 222$, $b = 318$, $c = 406$: find A.

$\log 473 = 2{\cdot}6748611$, $\log 251 = 2{\cdot}3996737$,
$\log 406 = 2{\cdot}6085260$, $L \cos 16^\circ\ 28' = 9{\cdot}9818117$,
$\log 318 = 2{\cdot}5024271$, $L \cos 16^\circ\ 29' = 9{\cdot}9817744$.

18. Given $a = 11$, $b = 13$, $c = 16$: find all the angles.

$\log 2 = {\cdot}3010300$, $L \cot 21^\circ\ 31' = 10{\cdot}4042321$,
$\log 3 = {\cdot}4771213$, $L \cot 21^\circ\ 32' = 10{\cdot}4038620$,
$\log 7 = {\cdot}8450980$, $L \cot 41^\circ\ 35' = 10{\cdot}0519190$,
$L \cot 41^\circ\ 36' = 10{\cdot}0516645$.

19. Given $a = 25$, $b = 26$, $c = 27$: find all the angles.

$\log 35 = 1{\cdot}5440680$, $L \tan 28^\circ\ 7'\ 30'' = 9{\cdot}7279568$,
$\log 3 = {\cdot}4771213$, $L \tan 28^\circ\ 7'\ 40'' = 9{\cdot}7280074$,
$L \tan 31^\circ\ 56'\ 50'' = 9{\cdot}7948986$,
$L \tan 31^\circ\ 57' = 9{\cdot}7949455$.

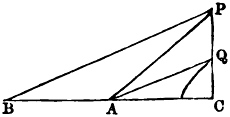
20. If the angles of a plane triangle be in Arithmetical Progression, and the greatest side be to the least as 5 is to 4, find all the angles.

$\log 3 = {\cdot}4771213$, $L \tan 10^\circ\ 53'\ 30'' = 9{\cdot}2842475$,
$L \tan 10^\circ\ 53'\ 40'' = 9{\cdot}2843610$.

XII. *Heights and Distances.*

114. We have already in Arts. 38...45 considered some of the cases which properly belong to the present Chapter; and the student is advised to read those Articles again before proceeding to the additional investigations which we shall now give.

115. *To find the height of an inaccessible object placed on a hill.*



Let P be the top of the object, PQ its height; let A and B be two points in a horizontal plane, such that P, Q, A, B are all in the same vertical plane. Let PQ produced meet BA produced at C. At A observe the angles PAC and QAC; at B observe the angle PBC; and measure AB.

The angle $APB = PAC - PBC$, and is therefore known.

Then $$\frac{AP}{AB} = \frac{\sin PBA}{\sin APB};$$

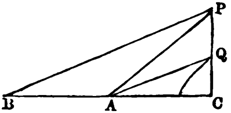
therefore $$AP = \frac{AB \sin PBA}{\sin APB}.$$

The angle $PAQ = PAC - QAC$, and is therefore known. The angle $PQA = QCA + QAC = 90^\circ + QAC$, and is therefore known.

Then $$\frac{PQ}{PA} = \frac{\sin PAQ}{\sin PQA};$$

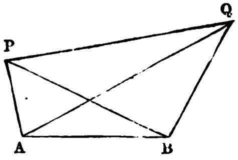
therefore $$PQ = \frac{PA \sin PAQ}{\sin PQA};$$

thus PQ is known.



Also CQ may be found, if required. For we may find AC from the triangle PAC, and then CQ from the triangle ACQ. Or we may find PC from the triangle PAC, and then by subtracting PQ we have CQ.

116. *To find the distance between two inaccessible points which are visible from two accessible points.*



Let P and Q be the inaccessible points; A and B the accessible points, from which P and Q are visible.

We will first suppose all the four points to be in the same plane.

At A observe the angles PAQ and QAB; at B observe the angles PBA and QBA; and measure AB.

Then in the triangle APB, the side AB and the angles PAB and PBA are known; thus PA can be found. Again, in the triangle ABQ, the side AB and the angles QAB and ABQ are known; thus AQ can be found. Lastly, in the triangle PAQ, the sides AP and AQ and the angle PAQ are known; thus PQ can be found.

If the four points are not all in the same plane the only difference is that we must observe the angle PAB as well as the angles PAQ and QAB; for now the angle PAB will not be equal to the sum of the angles PAQ and QAB.

117. Suppose that in the figure of the preceding Article we *know* PQ and wish to determine AB without measurement, having observed the same angles as before. We may adopt the following method:

We have
$$AP = \frac{AB \sin PBA}{\sin APB},$$
$$AQ = \frac{AB \sin QBA}{\sin AQB};$$
and, by Art. 106,
$$PQ^2 = AP^2 + AQ^2 - 2AP \cdot AQ \cos PAQ:$$
therefore
$$PQ^2 = AB^2 \left\{ \frac{\sin^2 PBA}{\sin^2 APB} + \frac{\sin^2 QBA}{\sin^2 AQB} - \frac{2 \sin PBA \sin QBA \cos PAQ}{\sin APB \sin AQB} \right\}.$$

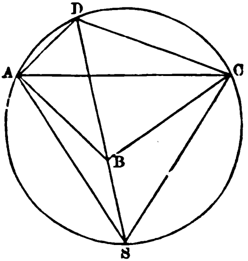
From this formula we can determine AB, since PQ and all the angles which occur are known; but the formula is not suited for the application of logarithms. Hence the following method is practically better:

Suppose a second figure *similar* to $ABQP$ and denote it by $abqp$; then we shall have
$$\frac{AB}{PQ} = \frac{ab}{pq}.$$

Hence *assume* any value for ab, and calculate pq from the observed angles by the method of Art. 116; then find AB from the relation just given.

118. The problem in the next Article is of practical importance for military and other purposes. It may be necessary to know the distance of a station from some inaccessible objects; by the aid of a good map the distances of three prominent objects from each other may be ascertained, and then by observing angles and by calculation we can determine the distance of a station from these objects. This we shall now shew.

119. *Given the distances of three points from each other, to find their distances from a station in the same plane with them.*

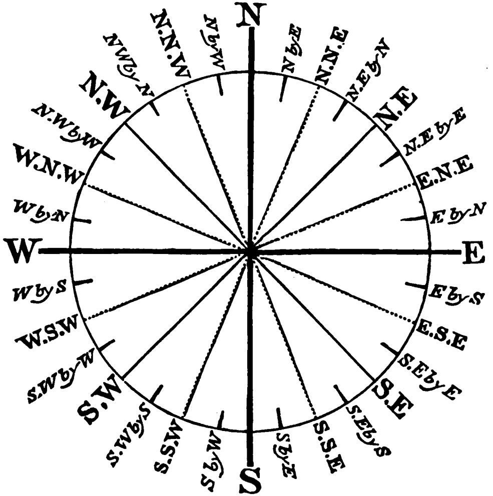


Let A, B, C denote the three points, so that AB, BC and CA are all known. Let the angles which AB and BC subtend at the station be observed. At the point A make the angle CAD equal to the observed angle at the station subtended by BC; and at the point C make the angle ACD equal to the observed angle at the station subtended by AB. Describe a circle round the triangle ACD, and produce DB to meet the circumference again at S. Then S will denote the station.

For the angle $BSC =$ the angle CAD, and the angle $BSA =$ the angle ACD; by Euclid III. 21. Hence the angles subtended at S by BA and BC are equal to the angles observed at the station; and therefore S denotes the station. Thus SA, SB and SC represent the three required distances: and we shall now shew how they may be calculated.

In the triangle ADC the side AC and the angles CAD and ACD are known; thus AD can be found. In the triangle ABC all the sides are known; thus the angle BAC can be found. Hence the angle BAD is known. In the triangle BAD the sides AB and AD, and the angle BAD are known; thus the other angles can be found. Hence the angle ABS is known. In the triangle ABS the side AB and the angles ASB and ABS are known; thus the sides SA and SB can be found. In the triangle ASC the sides AC and AS and the angle ASC are known; thus the side SC can be found.

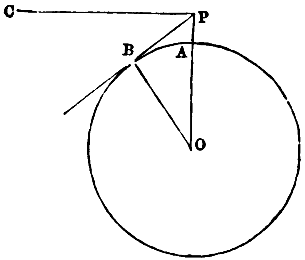
120. *Mariner's Compass.* In some problems the directions of straight lines are indicated by the terms used in the *Mariner's Compass*, which we will now explain.



The circumference of a circle is divided into 32 equal parts. Thus there are 32 *Points*, which have the names indicated in the figure, where N. S. E. and W. stand for North, South, East, and West respectively. The angle between two adjacent points is $\frac{360^0}{32}$, that is $11\frac{1}{4}^0$.

Hence we can readily determine the angle between straight lines drawn in directions which are indicated by these names. For example, if one object appear N.E. of an observer and another E.S.E. the angle between straight lines drawn from the observer towards these two objects is $6 \times 11\frac{1}{4}^{\circ}$, that is $67\frac{1}{2}^{\circ}$.

121. *Dip of the Horizon.* The earth is known to be very nearly spherical in shape. Suppose that from any point above the earth's surface straight lines are drawn to touch the earth. Then neglecting the inequalities of the earth's surface these straight lines will touch the earth at points which lie in the circumference of a circle. This circle bounds the portion of the earth's surface which is visible to a spectator at the point from which the straight lines are drawn, and is called the *terrestrial horizon* of the spectator.



Thus let O denote the centre of the earth, P a point above the surface, PB a straight line drawn from P to touch the surface at B: then B is a point on the horizon. Draw PC at right angles to PO in the same plane as B and O; then the angle CPB is called the *dip of the horizon* at P.

Let OP cut the surface of the earth at A. Let the angle BPC be denoted by θ. The following relations hold:

The angle $BOP = BPC = \theta$,

$$OP = OB \sec\theta,$$

$$AP = OP - OA = OA(\sec\theta - 1) = \frac{OA(1-\cos\theta)}{\cos\theta},$$

$$PB = OB\tan\theta = \frac{AP\cos\theta}{1-\cos\theta}\tan\theta = \frac{AP\sin\theta}{1-\cos\theta}$$

$$= AP\cot\frac{\theta}{2}, \text{ by Art. 35.}$$

These relations are useful in solving problems relating to the dip of the horizon, and the distance of the horizon.

A formula which depends only on Geometry, may be conveniently noticed here. We know by Euclid III. 36 that

$$PB^2 = PA(PA + 2OA);$$

now in all cases which can occur in practice PA is extremely small compared with $2OA$, so that we have very approximately

$$PB^2 = 2PA.OA.$$

The number of miles in OA is very approximately 3960; let n denote the number of miles in PB: then the number of *feet* in PA

$$= \frac{(n \times 5280)^2}{2 \times 3960 \times 5280} = \frac{528n^2}{2 \times 396} = \frac{2}{3} n^2.$$

Thus if $n = 1$ we have $h = \frac{2}{3}$ of a foot $= 8$ inches very nearly; if $n = 2$ we have $h = \frac{2}{3}$ of 4 feet $= 32$ inches very nearly.

The surface of still water though apparently plane is really curved; and it appears from the above calculation that an object less than 8 inches above the surface of still water will be invisible to an eye on the surface at the distance of a mile.

122. The term *angle of depression* sometimes occurs in problems of heights and distances: in such cases the spectator is supposed to be at a point P above the surface of the earth, and the angle of depression of any point is the angle between the straight line PC and the straight line drawn from P to the point, which is somewhere *below* PC. Similarly to a spectator at P the angle of elevation of a point is the angle between PC and the straight line drawn from P to the point which is somewhere *above* PC.

EXAMPLES. XII.

1. The angles of depression of the top and bottom of a column observed from a tower 108 feet high are 30^0 and 60^0 respectively: find the height of the column.

2. At the foot of a mountain the elevation of its summit is found to be 45^0. After ascending for one mile, at a slope of 15^0, towards the summit, its elevation is found to be 60^0. Find the height of the mountain.

3. A and B are two stations on a hill side; the inclination of the hill to the horizon is 30^0; the distance between A and B is 500 yards. C is the summit of another hill in the same vertical plane as A and B, on a level with A, but at B its elevation above the horizon is 15^0. Find the distance between A and C.

4. A ship which is known to be sailing due East at 12 miles an hour, was observed at noon to be 15^0 to the East of South; at 1 h. 30 m. after noon the ship was seen in the South East: determine the distance of the ship when first seen.

5. A person wishing to know the distance of a point C measures a straight line AB, and finds it to be 100 yards; he observes that the angles BAC and ABC are respectively $53^0\,20'$ and $59^0\,30'$: determine the distance of C from A.

$L\sin 59^0\,30' = 9{\cdot}9353204,\qquad \log 93489 = 4{\cdot}9707605.$

$L\sin 67^0\,10' = 9{\cdot}9645602,$

6. A person standing on the bank of a river observes the elevation of the top of a tree on the opposite bank to be 51^0; and when he retires 30 feet from the river's edge he finds the elevation to be 46^0: find the breadth of the river.

$L \sin 46^\circ = 9{\cdot}8569341$, $\log 3 = {\cdot}4771213$,

$L \sin 39^\circ = 9{\cdot}7988718$, $\log 1{\cdot}55823 = {\cdot}1926316$.

$L \sin 5^\circ = 8{\cdot}9402960$,

7. From the top of a hill I observe two milestones on the level ground in a straght line before me, and I find their angles of depression to be respectively 5° and 15°: find the height of the hill.

$L \sin 5^\circ = 8{\cdot}9402960$, $\log 12990 = 4{\cdot}1136092$,

$L \sin 10^\circ = 9{\cdot}2396702$, $\log 12991 = 4{\cdot}1136426$.

$L \sin 15^\circ = 9{\cdot}4129962$,

8. A tower is situated on the top of a hill whose angle of inclination to the horizon is 30°; the angle subtended by the tower at the foot of the hill is found by an observer to be 15°; and on ascending 485 feet up the hill the tower is found to subtend an angle of 30°: find the height of the tower and the distance of its base from the foot of the hill.

$\log 3 = {\cdot}4771213$, $\log 280{\cdot}015 = 2{\cdot}4471813$.

$\log 485 = 2{\cdot}6857417$,

9. *ABCD* is a rectangular piece of water the dimensions of which are required, but on account of the nature of the ground the only measures which can be taken are the angles that *BC* subtends at *A*, and at a point *P* which is 220 feet from *A* in *BA* produced, the former being 71° and the latter 55°. Find the length and breadth of the rectangle.

$L \sin 16^\circ = 9{\cdot}4403381$, $\log 2{\cdot}2 = {\cdot}3424227$,

$L \sin 55^\circ = 9{\cdot}9133645$, $\log 2{\cdot}12858 = {\cdot}3280900$,

$L \sin 71^\circ = 9{\cdot}9756701$, $\log 6{\cdot}18186 = {\cdot}7911193$.

$L \cos 71^\circ = 9{\cdot}5126419$,

10. In a survey it is found necessary to continue a straight line *AB* past an obstacle, which, from its height, hinders the view of the parts beyond. A straight line *BD*

is therefore measured at right angles to AB, and from the point D straight lines DP, DQ are drawn which clear the obstacle; the angles BDP and BDQ are found to contain 41° and 68° respectively, the distance BD being 180 yards. Determine the lengths which must be set off along DP and DQ to ensure that PQ shall be in the prolongation of AB.

$L \cos 41^\circ = 9{\cdot}8777799$, $\log 1{\cdot}8 = {\cdot}2552725$,

$L \sin 22^\circ = 9{\cdot}5735754$, $\log 2{\cdot}38502 = {\cdot}3774920$,

$\log 4{\cdot}80504 = {\cdot}6816970$.

11. A ship of a blockading squadron lies 4 miles to the South of a harbour, and observes that a ship leaves the harbour in a direction E. 30° S. If the blockading ship sails 12 miles an hour, in what direction must she go so as to cross the course of the other ship in three quarters of an hour?

$L \sin 22^\circ\, 38' = 9{\cdot}5852716$, $\log 2 = {\cdot}3010300$,

$L \sin 22^\circ\, 39' = 9{\cdot}5855745$, $\log 3 = {\cdot}4771213$.

12. The courses of two ships are N. and E. and their rates of sailing are equal; the bearing of the former with respect to the latter was E.N.E., but after each had sailed four miles the bearing was N.N.W.: determine the distance between the ships at the time of the first observation.

13. While sailing S.W. I observe two ships at anchor, one N.N.W., and the other W.N.W. After running 5 miles these ships are seen N. and N.W. respectively. Determine their bearing and distance from each other.

14. A ship sailing out of harbour is watched by an observer from the shore; and at the instant she disappears below the horizon he ascends to a height of 20 feet, and thus retains her in sight 40 minutes longer. Find the rate at which the ship is sailing, assuming the Earth to be a sphere of 4000 miles radius.

15. An observer from the deck of a ship 20 feet above the level of the sea can just see the top of a distant lighthouse, and on ascending to the masthead, which is 60 feet above the deck, he sees the door which he knows to be one-fourth of the height of the lighthouse above the level of the sea. Find his distance from the lighthouse and its height, assuming the earth to be a sphere of 4000 miles radius.

16. Two observers in the same horizontal plane stationed at a distance of 200 yards from each other observed the altitude and bearing of the top of a tower; to one of them the altitude was 60^0 bearing S. W., and to the other the altitude was 45^0 bearing W.: find the height of the tower.

17. A flag staff a feet high on the top of a tower is seen from a certain point in the horizontal plane on which the tower stands to subtend at the eye of the observer an equal angle (A) with the tower itself. Shew that the height of the tower $=a \cos 2A$.

18. A person walks a yards from A to E along AB the side of a triangle ACB, and observes the angle $AEC=\theta$; again he walks b yards from B to F along the side BA and observes the angle CFB also $=\theta$; the whole distance AB being c, solve the triangle ABC.

19. The angular elevations of the top of a tower are observed to be a, β, γ at the stations A, B, C respectively; the stations are in a horizontal straight line, the direction of which does not pass through the tower; the distance from A to B is p, and from B to C is q, and B is between A and C: shew that $p \cot^2 \gamma + q \cot^2 a$ is greater than $(p+q) \cot^2 \beta$.

20. The altitude of a cloud was observed to be a, and that of the sun in the same direction to be β, and the distance of the shadow of the cloud from the station of the observer was found to be c feet: shew that the height of the cloud was $\frac{c \sin a \sin \beta}{\sin (a-\beta)}$ feet.

In solving the following Examples the Tables will be required:

21. From a window it is observed that the angle of elevation of the top of a house on the opposite side of the street is 29°, and the angle of depression of the bottom of the house is 56°; determine the height of the house, supposing the breadth of the street to be 80 feet.

22. The elevations of two mountains in the same straight line with an observer are 9° 30′ and 18° 20′: on approaching four miles nearer they have both an elevation of 37°. Find the heights of the mountains in yards.

23. P and Q are two inaccessible objects; a straight line AB in the same plane as P and Q is measured and found to be 280 yards; the angle PAB is 95°, the angle QAB is 47° 30′, the angle QBA is 110°, and the angle PBA is 52° 20′. Determine the length of PQ.

24. A, B, C are three objects at known distances apart; namely $AB=1056$ yards, $AC=924$ yards, and $CB=1716$ yards. An observer places himself at a station S from which C appears directly in front of A, and observes the angle CSB to be 14° 24′. Find the distance CS.

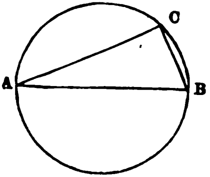
25. A, B, C are three objects at known distances apart; namely $AB=320$ yards, $AC=600$ yards, $BC=435$ yards. From a station S it is observed that $ASB=15°$, and $BSC=30°$. Find the distances of S from A, B, and C; the point B being nearest to S, and the angle ASC being the sum of ASB and BSC.

XIII. *Geometrical Solutions.*

123. The present Chapter will consist of Geometrical Solutions of some Trigonometrical Problems.

124. *To construct an angle with a given sine or cosine.*

Suppose we require an angle the *sine* of which is a given quantity a.



Describe a circle with unity for its diameter, and draw any diameter AB of this circle. With centre B, and radius equal to a, describe a circle; let C be one of the points where the circumferences of the two circles intersect: join AC and BC.

Then ACB is a right angle, by Euclid III. 31; and therefore the sine of BAC is $\frac{BC}{AB}$, that is a. Therefore BAC is such an angle as is required.

If we require an angle the *cosine* of which is a given quantity a, then the same construction may be made; and ABC will be such an angle as is required.

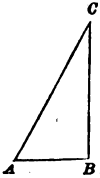
125. If an angle is required to have a given *cosecant*, then since the cosecant is the reciprocal of the sine the angle must have a known sine, and therefore may be found by the preceding Article.

Similarly, if an angle is required to have a given *secant* or a given *versed sine*, the angle must have a known cosine, and therefore may be found by the preceding Article.

126. *To construct an angle with a given tangent or cotangent.*

Suppose we require an angle the tangent of which is a given quantity a.

If the tangent of an angle is a the cosine is $\frac{1}{\sqrt{(a^2+1)}}$, and the sine is $\frac{a}{\sqrt{(a^2+1)}}$. Therefore since these quantities are known, we can construct the angle by Art. 124. Or we may proceed independently thus:



Take a straight line AB the length of which is unity; draw BC at right angles to AB and equal to a, and join CA.

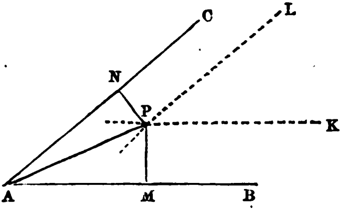
Then the tangent of BAC is $\frac{BC}{AB}$, that is a. Therefore BAC is such an angle as is required.

If we require an angle the *cotangent* of which is a given quantity a, then the same construction may be made; and ACB will be such an angle as is required.

127. *To divide a given angle into two parts which shall have their sines in a given ratio.*

Let BAC be the given angle, and let it be required to divide it into two parts, such that the sine of one part may be to the sine of the other as m is to n.

Draw a straight line KP parallel to AB, and at a distance m from it; draw a straight line LP parallel to AC, and at a distance n from it. Let P be the point of intersection of these straight lines; join AP. Then AP shall divide the angle BAC in the required manner.



For draw PM perpendicular to AB, and PN perpendicular to AC. Then

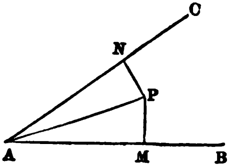
$$\sin PAB = \frac{PM}{AP} = \frac{m}{AP};$$

$$\sin PAC = \frac{PN}{AP} = \frac{n}{AP};$$

therefore

$$\frac{\sin PAB}{\sin PAC} = \frac{m}{AP} \div \frac{n}{AP} = \frac{m}{AP} \times \frac{AP}{n} = \frac{m}{n}.$$

128. *To divide a given angle into two parts which shall have their cosines in a given ratio.*



Let BAC be the given angle, and let it be required to divide it into two parts, such that the cosine of one part may be to the cosine of the other as m is to n.

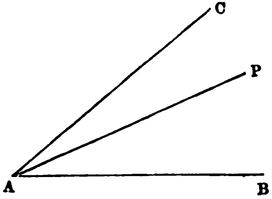
On AB take $AM=m$, and on AC take $AN=n$; draw MP at right angles to AB, and NP at right angles to AC. Let P be the point of intersection of these straight lines; join AP. Then AP shall divide the angle BAC in the required manner.

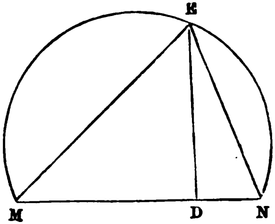
For
$$\cos PAB = \frac{AM}{AP} = \frac{m}{AP};$$
$$\cos PAC = \frac{AN}{AP} = \frac{n}{AP}:$$
therefore
$$\frac{\cos PAB}{\cos PAC} = \frac{m}{AP} \div \frac{n}{AP} = \frac{m}{AP} \times \frac{AP}{n} = \frac{m}{n}.$$

If the point P does not fall *within* the angle BAC, we conclude that the angle cannot be divided into two parts in the proposed manner.

129. *To divide a given angle into two parts which shall have their tangents in a given ratio.*

Let BAC be the given angle, and let it be required to divide it into two parts such that the tangent of one part may be to the tangent of the other as m is to n.





Take any straight line MN, and divide it at D so that DM may be to DN as m is to n. On MN describe a segment of a circle containing an angle equal to the angle BAC; draw DE at right angles to MN meeting the circumference at E. Join EM and EN. Then ED shall divide the angle MEN in the required manner.

For $\tan DEM = \dfrac{DM}{DE}$; $\tan DEN = \dfrac{DN}{DE}$;

therefore $$\frac{\tan DEM}{\tan DEN} = \frac{DM}{DE} \div \frac{DN}{DE} = \frac{DM}{DN} = \frac{m}{n}.$$

At the point A, in the straight line AB, make the angle PAB equal to the angle DEN; then PAC is equal to DEM: and AP divides the angle BAC in the manner required.

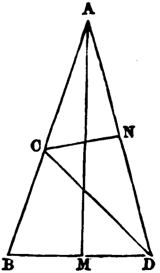
130. *To divide a given angle into two parts which shall have their cotangents in a given ratio.*

This may be solved as in the last Article. For with the notation there used we have

$$\cot DEM = \frac{DE}{DM}; \quad \cot DEN = \frac{DE}{DN};$$

therefore $$\frac{\cot DEN}{\cot DEM} = \frac{DE}{DN} \div \frac{DE}{DM} = \frac{DM}{DN} = \frac{m}{n}.$$

131. By the aid of a problem in Euclid we can find an expression for the Trigonometrical Ratios of an angle of 18°, as we will now shew.



In Euclid IV. 10 the following result is obtained:

ABD is a triangle in which $AB=AD$, and *C* is a point such that $AB \,.\, BC=AC^2$; and $AC=BD$: then each of the angles at *B* and *D* is double the angle at *A*.

Hence the angle at *A* is the fifth part of two right angles, and therefore contains 36°; and each of the angles at *B* and *D* contains 72°.

Draw *AM* perpendicular to *BD*; then the angle *BAM* contains 18°.

Since $AB \,.\, BC=AC^2$, we have

$$AB(AB-AC)=AC^2;$$

thus

$$\frac{AC^2}{AB^2}+\frac{AC}{AB}-1=0.$$

By solving the equation we obtain

$$\frac{AC}{AB}=\frac{-1\pm\sqrt{5}}{2},$$

and we must take the upper sign, for $\frac{AC}{AB}$ is a positive quantity. Thus

$$\frac{AC}{AB}=\frac{\sqrt{5}-1}{2}.$$

Now $\sin 18^\circ = \dfrac{BM}{AB} = \dfrac{1}{2}\dfrac{BD}{AB} = \dfrac{1}{2}\dfrac{AC}{AB} = \dfrac{\sqrt{5}-1}{4}$.

Hence we may deduce the values of the other Trigonometrical Ratios for an angle of 18°. For

$$\cos^2 18^\circ = 1 - \sin^2 18^\circ = 1 - \left(\frac{\sqrt{5}-1}{4}\right)^2$$

$$= 1 - \frac{6-2\sqrt{5}}{16} = \frac{10+2\sqrt{5}}{16};$$

therefore $$\cos 18^\circ = \frac{\sqrt{(10+2\sqrt{5})}}{4}.$$

And so on for the other Trigonometrical Ratios of 18°.

Again, draw *CN* perpendicular to *AD*.

Then $$\cos 36^\circ = \frac{AN}{AC};$$

and $AN = ND$, for the triangles *ACN* and *DCN* are equal in all respects, by the reasoning in Euclid IV. 10; thus

$$\cos 36^\circ = \frac{1}{2}\frac{AD}{AC} = \frac{1}{\sqrt{5}-1} = \frac{(\sqrt{5}+1)}{(\sqrt{5}-1)(\sqrt{5}+1)}$$

$$= \frac{\sqrt{5}+1}{4}.$$

Or we may obtain this result without recurring again to the figure. For, by Art. 35,

$$\cos 36^\circ = 1 - 2\sin^2 18^\circ = 1 - \frac{2(\sqrt{5}-1)^2}{16} = 1 - \frac{3-\sqrt{5}}{4}$$

$$= \frac{\sqrt{5}+1}{4}.$$

Then $\sin 36^\circ = \sqrt{(1-\cos^2 36^\circ)} = \dfrac{\sqrt{(10-2\sqrt{5})}}{4}$.

And so on for the other Trigonometrical Ratios of 36°.

Examples. XIII.

1. A certain angle is equal to four times its complement: determine the angle.

2. The sine of one angle is equal to the cosine of another; and the number of degrees in one is three-fifths of the number of grades in the other: determine the angles.

3. Construct an angle whose tangent is four times its sine.

4. Divide a given angle into two parts, so that the sine of one part may bear a given ratio to the cosine of the other part.

5. Shew that $\dfrac{\tan A + \tan B}{\cot A + \cot B} = \tan A \tan B$.

6. Shew that the area of any quadrilateral figure is equal to half the product of the two diagonals into the sine of the angle between them.

7. In the *ambiguous* case when a, b and A are given, shew that the difference of the squares of the two values of the third side is $4b \cos A \sqrt{(a^2 - b^2 \sin^2 A)}$.

8. If $\dfrac{\sin \alpha}{\sin \beta} = \sqrt{2}$, and $\dfrac{\tan \alpha}{\tan \beta} = \sqrt{3}$, find α and β.

9. If $\sin A \sin B = \sin \alpha \sin \beta$, $\cos A \cos B = \cos \alpha \cos \beta$

and $$\cos^2 A + \cos^2 B - \cos^2 \gamma = 1;$$

then $$\sin^2 \alpha + \sin^2 \beta = \sin^2 \gamma.$$

10. If $\cos x = n \sin \alpha$, and $\cot x = \sin \alpha \cot \beta$, then

$$\cos^2 \beta = \frac{n^2}{1 + n^2 \cos^2 \alpha}.$$

11. If $\operatorname{cosec}^2\theta = m\tan\theta$ and $\sec^2\theta = n\cot\theta$,

then $$(mn)^3 = (\sqrt{m} + \sqrt{n})^4.$$

12. A rock is observed from the deck of a vessel to bear N.N.W.; and after the vessel has sailed 10 miles in the direction E.N.E. the same rock bears due W. Find the distance of the rock from the observer at the time of each observation.

13. AC is horizontal; AB, CD are two vertical towers. The angle of elevation of D observed at A is α, and observed at B is β; and $AB = h$. Find AC and CD.

14. If $a = 10$, $b = 8$, $c = 12$, find the angles:

$$\log 2 = \cdot 3010300, \quad L\cos 41^\circ 24' = 9\cdot 8751256,$$

$$\log 3 = \cdot 4771213, \quad L\cos 41^\circ 25' = 9\cdot 8750142.$$

15. If $a = 1$, $b = 7$, $c = \sqrt{(56)}$, find the angles:

$$\log 2 = \cdot 3010300, \quad L\sin 32^\circ 18' 40'' = 9\cdot 7279609,$$

$$\log 7 = \cdot 8450980, \quad L\sin 32^\circ 18' 50'' = 9\cdot 7279942.$$

16. A straight line CD subtends an angle α at a point A and also at a point B; and CA is at right angles to BD: shew that $AB = CD\cot\alpha$.

17. If $\cos\theta = \tan\lambda\cot\alpha$, $\cos\phi = \tan\lambda\cot\beta$, and

$$\sec\theta\sec\phi = \sec\lambda\tan\theta\tan\phi - \tan\alpha\tan\beta;$$

then $$\cos^2\lambda = \cos^2\alpha\cos^2\beta.$$

18. Find $\sin A$ and $\sin B$ from the equations

$$a\sin^2 A + b\sin^2 B = c, \quad a\sin 2A = b\sin 2B.$$

XIV. *Properties of Triangles.*

132. The present Chapter will contain some miscellaneous propositions chiefly relating to properties of triangles.

133. *To find an expression for the area of a triangle in terms of the sides.*

If b and c denote two sides of the triangle, and A the included angle, the area is $\frac{1}{2}bc\sin A$ by Art. 47. Now by Art. 108

$$\sin A = \frac{2}{bc}\sqrt{\{s(s-a)(s-b)(s-c)\}};$$

therefore the area of the triangle $=\sqrt{\{s(s-a)(s-b)(s-c)\}}$.

By the same Article, we have the area of the triangle

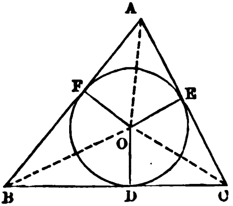
$$=\tfrac{1}{4}\sqrt{(2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4)}.$$

It is usual to denote by S the expression

$$\sqrt{\{s(s-a)(s-b)(s-c)\}},$$

or
$$\tfrac{1}{4}\sqrt{(2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4)}.$$

134. *To find the radius of the circle inscribed in a triangle.*



Let ABC be a triangle; and let O be the centre of the circle inscribed in the triangle, and touching the sides at the points D, E, and F. Join OD, OE, and OF.

Let r denote the radius of the circle. Then

$$\text{the area of the triangle } BOC = \tfrac{1}{2}BC.OD = \frac{ar}{2};$$

$$\text{the area of the triangle } COA = \tfrac{1}{2}CA.OE = \frac{br}{2};$$

$$\text{the area of the triangle } AOB = \tfrac{1}{2}AB.OF = \frac{cr}{2}:$$

therefore, by addition

$$\frac{a+b+c}{2}r = \text{the area of the triangle } ABC$$

$$= S, \text{ by Art. 133,}$$

that is $sr = S$.

Therefore, $$r = \frac{S}{s}.$$

The radius of the inscribed circle is thus equal to the *area of the triangle divided by half the sum of the sides;* and various expressions can be obtained for the radius by employing the various expressions already given for the area of the triangle.

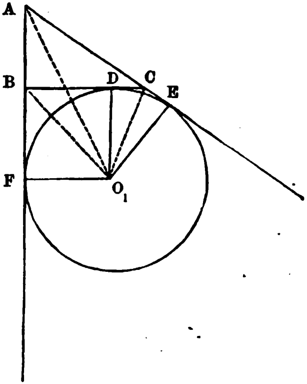
For example

$$r = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}};$$

$$r = \frac{ab\sin C}{2s} = \frac{ab\sin C}{a+b+c}.$$

135. *To find the radius of the circle which touches one side of a triangle and the other two sides produced.*

Let ABC be a triangle; and let O_1 be the centre of the circle which touches BC and the other sides produced. Let D, E, and F be the points of contact. Join O_1D, O_1E, and O_1F. Let r_1 denote the radius of the circle.



The quadrilateral O_1BAC may be divided into the two triangles O_1AB, O_1AC; therefore the area of this quadrilateral is $\frac{c}{2}r_1 + \frac{b}{2}r_1$. Again the same quadrilateral may be divided into the triangles O_1BC and ABC; therefore the area of this quadrilateral is $\frac{a}{2}r_1 + S$. Thus

$$\frac{c}{2}r_1 + \frac{b}{2}r_1 = \frac{a}{2}r_1 + S;$$

therefore $$\frac{c+b-a}{2}r_1=S,$$

that is $$(s-a)r_1=S.$$

Therefore $$r_1=\frac{S}{s-a}.$$

Similarly let r_2 denote the radius of the circle which touches CA and the other sides produced; and let r_3 denote the radius of the circle which touches AB and the other sides produced: then we shall find that

$$r_2=\frac{S}{s-b},\quad r_3=\frac{S}{s-c}.$$

A circle which touches one side of a triangle and the other sides produced is called an *escribed* circle.

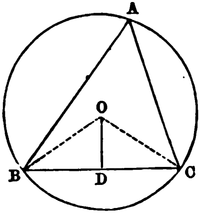
136. Examples and problems are frequently given respecting the inscribed and escribed circles of a triangle, which depend chiefly on the following geometrical facts:

In the figure of Art. 134 the straight lines OA, OB, OC bisect the angles A, B, C respectively, by Euclid IV. 4. In the figure of Art. 135 the straight line O_1A bisects the angle A, the straight line O_1B bisects the angle FBD, and the straight line O_1C bisects the angle ECD. See *notes on the fourth book of Euclid.* Thus the points A, O, and O_1 are on one straight line, namely that which bisects the angle A.

Let O_2 denote the centre of the escribed circle which touches CA, then the points O_1, C, and O_2 are on one straight line, namely that which bisects the angle BCE, which is the supplement of ACB.

The angle OBO_1 is a right angle.

137. *To find the radius of the circle described round a triangle.*



Let ABC be a triangle, and let O be the centre of the circle described round it. Draw OD perpendicular to BC; then BC is bisected at D, by Euclid IV. 5. Join OB and OC. Let R denote the radius of the circle.

The angle BOC is double of the angle BAC, by Euclid III. 20; therefore $BOD = A$.

And $$BD = R \sin A = \frac{a}{2};$$

therefore $$R = \frac{a}{2 \sin A}.$$

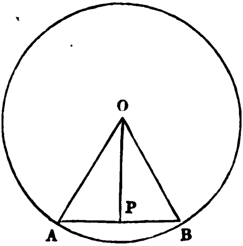
By Art. 108, $$\sin A = \frac{2S}{bc};$$

therefore $$R = \frac{abc}{4S}.$$

We see that $\dfrac{\sin A}{a} = \dfrac{1}{2R}$; thus we have, as stated in Art. 104,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{1}{2R}.$$

138. *To find the perimeter and the area of a regular polygon inscribed in a circle.*



Let O be the centre of a circle; let a regular polygon of n sides be inscribed in the circle, and let AB be one of the sides. Draw OP perpendicular to AB; and join OA, OB.

Let r denote the radius of the circle.

The angle $\qquad AOB = \dfrac{360^\circ}{n}$,

$$AB = 2AP = 2AO \sin AOP = 2r \sin \frac{180^\circ}{n}.$$

The area of the triangle $AOB = \frac{1}{2} AB \,.\, OP = AP \,.\, OP$

$$= AO \sin AOP \,.\, AO \cos AOP = r^2 \sin \frac{180^\circ}{n} \cos \frac{180^\circ}{n}.$$

Or, by Art. 47, the area of the triangle

$$= \tfrac{1}{2} AO \,.\, BO \sin AOB = \frac{r^2}{2} \sin \frac{360^\circ}{n}.$$

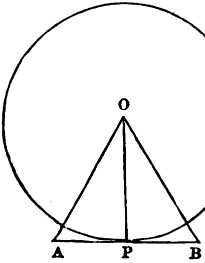
The perimeter of the regular polygon $= n \,.\, AB$

$$= 2nr \sin \frac{180^\circ}{n}.$$

The area of the regular polygon $= n \,.$ triangle AOB

$$= nr^2 \sin \frac{180^\circ}{n} \cos \frac{180^\circ}{n} = \frac{nr^2}{2} \sin \frac{360^\circ}{n}.$$

139. *To find the perimeter and the area of a regular polygon described about a circle.*



Let O be the centre of a circle; let a regular polygon of n sides be described about the circle, and let AB be one of the sides. Draw OP to the point of contact; then OP is perpendicular to AB, by Euclid III. 18. Join OA, OB.

Let R denote the radius of the circle.

The angle $$AOB = \frac{360^0}{n}.$$

$$AB = 2AP = 2OP \tan AOP = 2R \tan \frac{180^0}{n}.$$

The area of the triangle $AOB = \frac{1}{2} AB \,.\, OP = AP \,.\, OP$
$$= R^2 \tan \frac{180^0}{n}.$$

The perimeter of the regular polygon $= n \,.\, AB$
$$= 2nR \tan \frac{180^0}{n}.$$

The area of the regular polygon $= n \,.$ triangle AOB
$$= nR^2 \tan \frac{180^0}{n}.$$

140. The expressions found in the two preceding Articles may be applied in various ways.

Thus in Art. 138 we have

$$AB = 2r \sin \frac{180^\circ}{n};$$

hence $$r = \frac{AB}{2} \operatorname{cosec} \frac{180^\circ}{n}:$$

this determines r when AB is given.

Similarly, from Art. 139 we have

$$R = \frac{AB}{2} \cot \frac{180^\circ}{n}:$$

this determines R when AB is given.

Again, suppose a regular polygon of n sides to be inscribed in a circle, and another regular polygon of n sides to be described about the *same* circle: then

$$\frac{\text{side of inscribed polygon}}{\text{side of circumscribed polygon}} = \frac{\sin \frac{180^\circ}{n}}{\tan \frac{180^\circ}{n}} = \cos \frac{180^\circ}{n};$$

$$\frac{\text{area of inscribed polygon}}{\text{area of circumscribed polygon}} = \frac{\sin \frac{180^\circ}{n} \cos \frac{180^\circ}{n}}{\tan \frac{180^\circ}{n}}$$

$$= \cos^2 \frac{180^\circ}{n}.$$

The second of these two ratios might also be deduced from the first by Euclid VI. 20.

EXAMPLES. XIV.

1. Shew from the figure in Art. 134 that

$$r(\cot\tfrac{1}{2}B+\cot\tfrac{1}{2}C)=a.$$

2. Shew from the figure in Art. 135 that

$$r_1(\tan\tfrac{1}{2}B+\tan\tfrac{1}{2}C)=a.$$

3. Find the radius of the circle inscribed in an equilateral triangle.

4. Find the radius of the circle described about an equilateral triangle.

5. The sides of a triangle are 68, 75, and 77; determine the area, the radius of the inscribed circle, and the radius of the circumscribed circle.

6. In a triangle $a=243$ yards, $b=324$ yards, $c=405$ yards: find the area.

7. Find the area of the triangle in which $a=1864$, $b=c=2796$.

$\log 2=\cdot3010300$, $\log 24568=4\cdot3903698$,
$\log 932=2\cdot9694159$, $\log 24569=4\cdot3903875$.

8. Find the area of the triangle in which $a=942$, $b=812$, $c=1270$.

$\log 7=\cdot8450980$, $\log 1512=3\cdot1795518$,
$\log 57=1\cdot7558749$, $\log 3\cdot82094=\cdot5821700$,
$\log 242=2\cdot3838154$.

9. Shew that in the figure of Art. 134

$$OA=\frac{c\sin\frac{1}{2}B}{\cos\frac{1}{2}C}.$$

10. Shew that in the figure of Art. 135

$$O_1A=\frac{c\cos\frac{1}{2}B}{\sin\frac{1}{2}C}, \qquad O_1B=\frac{a\cos\frac{1}{2}C}{\cos\frac{1}{2}A}.$$

11. Shew from the figure in Art. 134 that $AF=s-a$, and

$$r=(s-a)\tan\frac{A}{2}.$$

12. From the preceding result deduce the value of r in Art. 134.

13. Shew from the figure in Art. 135 that $AF=s$, and

$$r_1=s\tan\frac{A}{2}.$$

14. From the preceding result deduce the value of r_1 in Art. 135.

15. In the *ambiguous case*, when a, b and A are given shew that the circles circumscribing both triangles are equal in magnitude, and that the distance between their centres is

$$\sqrt{(a^2\operatorname{cosec}^2 A-b^2)}.$$

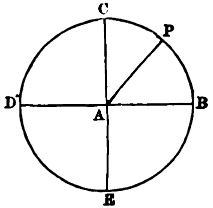
16. A tower is at right angles to a plane, and in the plane three points are found at which the tower subtends the same angle; these three points are distant from one another 123, 130, and 77 yards; also in passing from each one to the others in straight lines the greatest angle which the tower subtends at the eye is 45°. Shew that the height of the tower is $14\frac{5}{8}$ yards.

XV. *Angles greater than two right angles.*

141. In practical applications of Trigonometry it is not necessary to consider any angles except such as lie between zero and two right angles; and accordingly we have already explained the principles of the subject with sufficient generality for those who confine themselves to practical applications. But when Trigonometry is studied as a branch of theoretical mathematics it is found convenient to extend the notion of an angle; and this extension we shall now consider.

142. *Angles may be of any magnitude.*

Let BAD be any straight line, CAE a straight line at right angles to BAD. Suppose a straight line AP to revolve round one end A, starting from the position AB. When AP coincides in direction with AC, the angle which has been described is a right angle; when AP coincides in direction with AD, the angle which has been described is two right angles; when AP coincides in direction with AE, the angle which has been described is three right angles; when AP coincides in direction with AB, the angle which has been described is four right angles. Then as AP proceeds through a second revolution, the angle described will be greater than four right angles. Thus if AP be situated midway between AB and AC, the angle

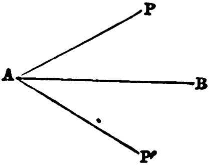


between AB and AP will be half a right angle if AP be supposed in its *first* revolution; the angle will be four right angles and a half if AP be supposed in its *second* revolution; the angle will be eight right angles and a half if AP be supposed in its *third* revolution; and so on.

143. The straight lines CAE and BAD form by their intersection four right angles; these are called *quadrants:* BAC is called the *first quadrant*, CAD the *second quadrant*, DAE the *third quadrant*, and EAB the *fourth quadrant.* Now suppose any angle formed by the fixed straight line AB and the moveable straight line AP; if AP is situated in the first quadrant, the angle BAP is said to be in the first quadrant; if AP is situated in the second quadrant, the angle BAP is said to be in the second quadrant; and so on.

144. *Angles may be negative.*

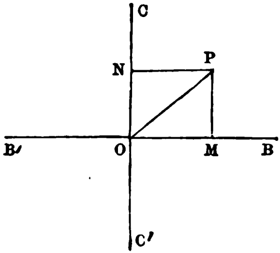
By a convention similar to that in Art. 91, we distinguish angles measured in one direction from angles measured in the opposite direction. Let a straight line start from the position AB, and by revolving in one direction



round A trace out the angle PAB, and let this angle be denoted by a *positive* number; then if the straight line start from AB, and by revolving round A in the opposite direction trace out the angle $P'AB$, this angle may be denoted by a *negative* number. If, for example, each of the angles BAP and BAP' be one-third of a right angle, and we denote the former by 30°, the latter will be denoted by -30°.

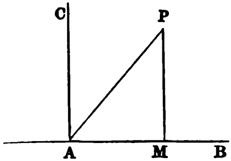
145. In Art. 91 we distinguished by means of positive and negative numbers between the two directions in which distances may be measured along a fixed straight line from a fixed point. In like manner we may distinguish between the two directions in which distances may be measured along a second straight line at right angles to the former.

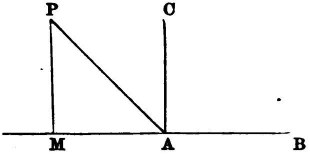
Let BOB', COC' be two straight lines which cut at right angles. Let P be any point in the plane which contains these two straight lines. The position of P will be known if we know the distance of P from each of the straight lines BB' and CC', and also know on *which side* of each of these straight lines P is situated. Draw PM and PN perpendicular to the straight lines BB' and CC' respectively. We shall adopt the following conventions: the distance NO or PM will be denoted by a *positive* number when P is *above* the straight line BB', and by a *negative* number when P is *below* the straight line BB'; the distance MO or PN will be denoted by a *positive* number when P is to the *right* of CC' and by a *negative* number when P is to the *left* of CC'.

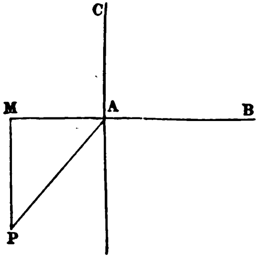


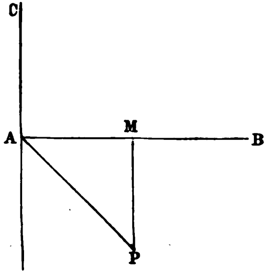
146. *General definitions of the Trigonometrical Ratios.*

We can now give general definitions of the Trigonometrical Ratios which will apply to angles of any magnitude.









Let AB, AC be two straight lines at right angles; let a straight line turn round the point A from AB towards AC and come into any position AP; draw PM perpendicular to AB or to AB produced through A. Then consider AP as always positive; consider AM as positive or negative according as M is on the same side of AC as B is, or on the opposite side; and consider PM as positive or negative according as P is on the same side of AB as C is, or on the opposite side. Let the angle PAB be denoted by A; then the Trigonometrical Ratios of A are thus defined:

$$\sin A = \frac{PM}{AP}, \quad \tan A = \frac{PM}{AM}, \quad \sec A = \frac{AP}{AM},$$

$$\cos A = \frac{AM}{AP}, \quad \cot A = \frac{AM}{PM}, \quad \operatorname{cosec} A = \frac{AP}{PM},$$

$$\operatorname{vers} A = 1 - \cos A, \qquad \operatorname{covers} A = 1 - \sin A.$$

147. We have therefore Trigonometrical Ratios for any *positive* angle whatever may be its magnitude; and we have also Trigonometrical Ratios for any *negative* angle by adopting the convention that the Trigonometrical Ratios for any negative angle shall be the same as they would be for what we may call the *corresponding* positive angle. Thus, for example, in the last figure we may consider BAP as a negative angle, the magnitude of which is -60°; then the Trigonometrical Ratios will be the same as for the angle formed by turning the moveable line AP in the positive direction until it reaches the position which it has in the figure; so that the Trigonometrical Ratios for the angle -60° will be the same as for the angle $360^\circ - 60^\circ$.

148. It follows immediately from the definitions that if two angles differ by four right angles or by any multiple of four right angles, the Trigonometrical Ratios of the two angles are the same.

149. The relations established in Arts. 19—23 between the Trigonometrical Ratios of angles not exceeding a right angle will now be seen to hold universally between the Trigonometrical Ratios of angles of any magnitude.

It will be sufficient for the student to satisfy himself that the following relations hold universally, as from these the others can be deduced:

$$\tan A = \frac{\sin A}{\cos A}, \qquad \cot A = \frac{\cos A}{\sin A},$$

$$\sec A = \frac{1}{\cos A}, \qquad \operatorname{cosec} A = \frac{1}{\sin A},$$

$$\sin^2 A + \cos^2 A = 1.$$

EXAMPLES. XV.

Find all the angles between 0 and 360° which satisfy the following twelve equations:

1. $\sin^2 A = \frac{1}{4}$.   2. $\cos^2 A = \frac{1}{2}$.

3. $\tan^2 A = 3$.   4. $\sin 2A = \frac{\sqrt{3}}{2}$.

5. $\cos 2A = -\frac{1}{2}$.   6. $\tan 2A = 1$.

7. $2\cos^2 A + \sin A = 1$.   8. $2\sin^2 A = 3(1 + \cos A)$.

9. $\tan^2 A - 4\tan A + 1 = 0$.   10. $\tan A - \cot A = 2$.

11. $\sin A = 1 - \cos 2A$.   12. $\cos A = 1 + \cos 2A$.

13. In any triangle, shew that

$$\frac{\text{vers}\, A}{\text{vers}\, B} = \frac{a(a+c-b)}{b(b+c-a)}.$$

14. In any triangle, shew that

$$\frac{\cot\frac{B}{2} + \cot\frac{C}{2}}{\cot\frac{A}{2}} = \frac{2a}{b+c-a}.$$

15. If a point be taken within a triangle so that the sides subtend equal angles at it, and α, β, γ be the distances of this point from the angular points of the triangle, shew that

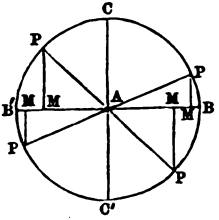
$$a^2 + b^2 + c^2 = 2(\alpha^2 + \beta^2 + \gamma^2) + \frac{4 \text{ area of the triangle}}{\sqrt{3}}.$$

16. If a triangle be divided into any two parts by a straight line drawn from one of the angles, shew that the radii of the circles described about these two triangles are in a ratio which is constant for all positions of the dividing straight line.

XVI. *Changes in the Ratios as the angle changes.*

150. In the present Chapter we shall trace the changes in magnitude and sign of the various Trigonometrical Ratios as the angle changes from zero to four right angles.

151. *To trace the changes in the sine of an angle as the angle varies.*



Let BAB' and CAC' be two straight lines at right angles, and suppose a straight line AP of constant length to turn round one end A from the fixed position AB, so that P traces out the circle $BCB'C'$. From any position of P draw PM perpendicular to BAB'; then

$$\sin PAB = \frac{PM}{AP}.$$

When AP coincides with AB the perpendicular PM vanishes; then when the angle is zero so also is its sine. While AP moves through the first quadrant PM is positive, and continually increases until AP coincides with AC, and then PM is equal to AP; thus as the angle increases from 0 to 90^0 the sine increases from 0 to 1. While AP moves through the second quadrant PM is positive and continually decreases until AP coincides with AB', and then PM vanishes; thus as the angle increases from 90^0 to 180^0 the sine diminishes from 1 to 0. While AP moves through the third quadrant PM is negative and increases *numerically* until AP coincides with AC'; thus as the angle increases from 180^0 to 270^0 the sine is *negative* and increases numerically from 0 to -1. While AP moves through the fourth quadrant PM is negative and decreases *numerically* until AP coincides with AB; thus as the angle increases from 270° to 360° the sine is *negative* and decreases numerically from -1 to 0.

152. *To trace the changes in the cosine of an angle as the angle varies.*

With the figure of Art. 151 we have $\cos PAB = \frac{AM}{AP}$.

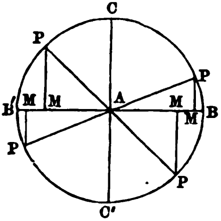
At first AP coincides with AB and then $AM = AP$, thus when the angle is zero the cosine is 1. While AP moves through the first quadrant AM is positive and continually decreases until AP coincides with AC and then AM vanishes; thus as the angle increases from 0 to 90° the cosine diminishes from 1 to 0. While AP moves through the second quadrant AM is negative and increases *numerically* until AP coincides with AB'; thus as the angle increases from 90° to 180° the cosine is negative and increases numerically from 0 to -1. While AP moves through the third quadrant AM is negative and decreases numerically until AP coincides with AC'; thus as the angle increases from 180° to 270° the cosine is *negative* and decreases numerically from -1 to 0. While AP moves through the fourth quadrant AM is positive and continually increases until AP coincides with AB; thus as the angle increases from 270° to 360° the cosine is positive and increases from 0 to 1.

153. *To trace the changes in the tangent of an angle as the angle varies.*

With the figure of Art. 151 we have $\tan PAB = \frac{PM}{AM}$.

At first AP coincides with AB, and then $AM = AB$; thus when the angle is zero so also is its tangent. While AP moves through the first quadrant PM and AM are positive; PM continually increases and AM continually decreases until AP coincides with AC; thus as the angle increases from 0 to 90° the tangent increases from 0 without limit, so that by taking an angle sufficiently near to 90° we can make the tangent as great as we please; this is usually expressed for the sake of abbreviation thus: *the tangent of* 90° *is infinite.* While AP moves through the second quadrant PM is positive and AM is negative;

PM continually decreases and *AM* increases *numerically* until *AP* coincides with *AB′*; thus as the angle increases from 90° to 180° the tangent is *negative* and decreases numerically from an indefinitely large value to zero. While *AP* moves through the third quadrant *PM* and *AM* are negative; *PM* increases *numerically* and *AM* decreases *numerically* until *AP* coincides with *AC′*; thus as the angle increases from 180° to 270°, the tangent is positive and increases from 0 without limit, so that by taking an angle sufficiently near to 270° we can make the tangent as great as we please; this as before is abbreviated thus: *the tangent of* 270° *is infinite.* While *AP* moves through the fourth quadrant *PM* is negative and *AM* is positive; *PM* continually decreases numerically, and *AM* increases until *AP* coincides with *AB*; thus as the angle increases from 270° to 360° the tangent is *negative* and decreases numerically from an indefinitely large value to zero.



Similarly the changes in the cotangent of an angle may be traced.

154. *To trace the changes in the secant of an angle as the angle varies.*

The changes in the secant of an angle may be traced by means of the figure in the same way as those of the sine, cosine, and tangent; or we may use the formula $\sec PAB = \frac{1}{\cos PAB}$, and infer the changes in the secant from the known changes in the cosine; we will adopt the latter method. As the angle increases from 0 to 90° the cosine diminishes from 1 to 0; thus the secant increases from 1 without limit, so we may say *the secant of* 90° *is infinite.* As the angle increases from 90° to 180° the cosine is *negative* and increases *numerically* from 0 to -1; thus the secant is *negative* and decreases *numerically* from an indefinitely large value to -1. As the angle increases from 180° to 270° the cosine is *negative* and decreases *numerically* from -1 to 0; thus the secant is *negative* and increases numerically from -1 to infinity. As the angle increases from 270° to 360° the cosine is positive and continually increases from 0 to 1; thus the secant is positive and diminishes from infinity to 1.

Similarly the changes in the cosecant of an angle may be traced.

155. *To trace the changes in the versed sine of an angle as the angle varies.*

Since $\text{vers}\,A = 1 - \cos A$, as the angle increases from 0 to 180° the versed sine increases from 0 to 2, and as the angle increases from 180° to 360° the versed sine diminishes from 2 to 0.

156. Thus we see that the sine and the cosine may have any value between -1 and $+1$; the tangent and the cotangent may have any value between $-\infty$ and $+\infty$; the secant and the cosecant may have any value between $-\infty$ and -1 and between $+1$ and $+\infty$. And it will be found on examination that no Trigonometrical Ratio changes its sign except when it passes through the value zero or the value infinity. The versed sine is always positive and may have any value between 0 and 2.

157. The student should carefully remember the signs of the Trigonometrical Ratios in the four quadrants: the following table exhibits them.

	1st	2nd	3rd	4th
sine	+	+	−	−
cosine	+	−	−	+
tangent	+	−	+	−

EXAMPLES. XVI.

Trace the changes in the sign and value of the following six expressions as A changes from 0 to 360°:

1. $\sin A + \cos A$.
2. $\sin A - \cos A$.
3. $\sin^2 A$.
4. $\cos^2 A - \sin^2 A$.
5. $\sin A + \operatorname{cosec} A$.
6. $\tan A + \sec A$.

7. Find the least value of $\tan^2 A + \cot^2 A$ when A varies.

8. Find the least value of $4\cos^2 A + \sec^2 A$ when A varies.

9. $ABCD$ is a quadrilateral figure which can be inscribed in a circle: shew that $AC \sin A = BD \sin B$.

10. Shew that the area of a regular hexagon inscribed in a circle is to the area of a regular octagon inscribed in the same circle as $3^{\frac{3}{2}}$ is to $2^{\frac{5}{2}}$.

11. The sides of a quadrilateral figure taken in order are 135, 180, 150 and 125 feet; and the angle contained by the first two is a right angle: determine the area of the figure.

12. ABC is a triangle, and D is the middle point of BC: shew that

$$AD^2 = \frac{1}{2}(b^2 + c^2) - \frac{a^2}{4}.$$

13. Shew that in any triangle the length of the perpendicular from A on the opposite side is equal to

$$\frac{b^2 \sin C + c^2 \sin B}{b + c}.$$

14. Shew that in any triangle

$$b^2 \sin 2A + a^2 \sin 2B = 2bc \sin A.$$

15. Shew that the perpendicular from an angle of a triangle on the opposite side is an Harmonic Mean between the radii of the adjacent escribed circles.

16. If $A+B+C=180^\circ$, shew that

$$\text{vers}\, A \;\text{vers}\,(B+C) = \sin A \sin(B+C).$$

With the notation of Arts. 134, 135, 136 establish the following results:

17. $$OO_1 = (r_1 - r)\,\text{cosec}\,\tfrac{1}{2}A = \frac{Sa\sqrt{(bc)}}{s(s-a)\sqrt{\{(s-b)(s-c)\}}}.$$

18. $$O_1O_2 = (r_1 + r_2)\sec\tfrac{1}{2}C = \frac{Sc\sqrt{(ab)}}{(s-a)(s-b)\sqrt{\{s(s-c)\}}}.$$

19. $$r = s\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}.$$

20. $$r = \frac{a-b}{\cot\frac{B}{2} - \cot\frac{A}{2}}.$$

21. $$OO_1^2 + O_2O_3^2 = \frac{a^2b^2c^2}{S^2}.$$

22. $$r\,.\,OO_1 = OB\,.\,OC.$$

23. $$OA\,.\,OO_1 = 4Rr.$$

24. Perpendiculars are drawn from the angles of a triangle on the opposite sides meeting at K: shew that

$$AK = 2R\cos A.$$

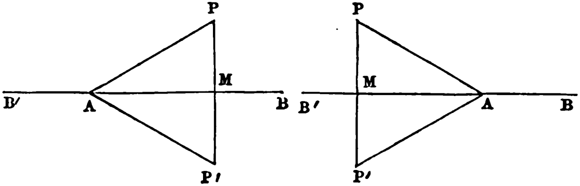
XVII. *Reduction of the angle.*

158. The object of the present Chapter is to shew that the Trigonometrical Ratios of any angle, positive or negative, can be expressed in terms of the Trigonometrical Ratios of an angle less than a right angle.

We shall require some preliminary propositions.

159. *To shew that*

$$\sin(-A) = -\sin A, \quad \textit{and} \quad \cos(-A) = \cos A.$$



Let PAB be any angle; draw PM perpendicular to the straight line BAB', and produce it to P' so that MP' may be equal in length to MP, and join AP'.

The angles $P'AB$ and PAB are numerically equal but are measured in opposite directions from AB; let PAB be denoted by A, then $P'AB$ will be denoted by $-A$. And

$$\sin A = \frac{PM}{AP}, \quad \sin(-A) = \frac{P'M}{AP'};$$

now $P'M$ is numerically equal to PM, but of opposite sign: thus

$$\sin(-A) = -\sin A.$$

Also $$\cos(-A) = \frac{AM}{AP'} = \frac{AM}{AP} = \cos A.$$

160. Hence we have:

$$\tan(-A)=\frac{\sin(-A)}{\cos(-A)}=\frac{-\sin A}{\cos A}=-\tan A;$$

$$\cot(-A)=\frac{\cos(-A)}{\sin(-A)}=\frac{\cos A}{-\sin A}=-\cot A;$$

$$\sec(-A)=\frac{1}{\cos(-A)}=\frac{1}{\cos A}=\sec A;$$

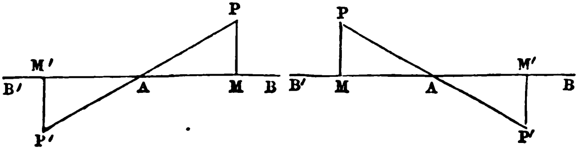
$$\operatorname{cosec}(-A)=\frac{1}{\sin(-A)}=\frac{1}{-\sin A}=-\operatorname{cosec} A;$$

$$\operatorname{vers}(-A)=1-\cos(-A)=1-\cos A=\operatorname{vers} A.$$

These results might also be obtained directly from the figures, as in Art. 159.

161. *To shew that*

$\sin(180^\circ+A)=-\sin A$, *and* $\cos(180^\circ+A)=-\cos A$.

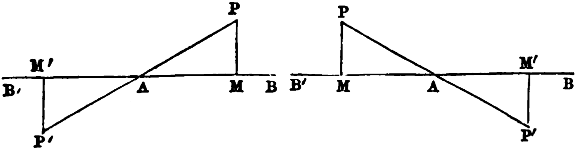


Let *PAB* be any angle; produce *PA* to *P'* so that *AP'* may be equal in length to *AP*: draw *PM* and *P'M'*, perpendicular to the straight line *BAB'*.

Let the angle *PAB* be denoted by *A*, then the angle *P'AB* measured in the same direction from *A* will be denoted by $180^\circ+A$. And

$$\sin A=\frac{PM}{AP},\quad \sin(180^\circ+A)=\frac{P'M'}{AP'},$$

$$\cos A=\frac{AM}{AP},\quad \cos(180^\circ+A)=\frac{AM'}{AP'}.$$



The triangles *PAM* and *P'AM'* are geometrically equal in all respects; thus *PM* and *P'M'* are numerically equal, but they are of opposite sign: also *AM* and *AM'* are numerically equal but of opposite sign. Thus

$$\sin(180^\circ + A) = -\sin A, \quad \cos(180^\circ + A) = -\cos A.$$

162. Hence we have:

$$\tan(180^\circ + A) = \frac{\sin(180^\circ + A)}{\cos(180^\circ + A)} = \frac{-\sin A}{-\cos A} = \tan A;$$

$$\cot(180^\circ + A) = \frac{\cos(180^\circ + A)}{\sin(180^\circ + A)} = \frac{-\cos A}{-\sin A} = \cot A;$$

$$\sec(180^\circ + A) = \frac{1}{\cos(180^\circ + A)} = \frac{1}{-\cos A} = -\sec A;$$

$$\operatorname{cosec}(180^\circ + A) = \frac{1}{\sin(180^\circ + A)} = \frac{1}{-\sin A} = -\operatorname{cosec} A;$$

$$\begin{aligned} \operatorname{vers}(180^\circ + A) = 1 - \cos(180^\circ + A) &= 1 + \cos A \\ &= 2 - \operatorname{vers} A. \end{aligned}$$

These results might also be obtained directly from the figures as in Art. 161.

163. We can now demonstrate the statement made in Art. 158.

By the formulæ in Arts. 159 and 160, we can express the Trigonometrical Ratios of a *negative* angle in terms of the Trigonometrical Ratios of a *positive* angle. Thus we need only consider positive angles.

By Art. 148 any multiple of four right angles may be rejected. Thus we need only consider angles less than four right angles.

By Arts. 161 and 162 we can express the Trigonometrical Ratios of an angle between two and four right angles in terms of the Trigonometrical Ratios of an angle less than two right angles.

By Art. 95 we can express the Trigonometrical Ratios of an angle between one and two right angles in terms of the Trigonometrical Ratios of an angle less than a right angle.

Thus the statement is demonstrated.

164. It will be observed that when we thus reduce the angle we can always express a Trigonometrical Ratio of any angle in terms of the *same* Trigonometrical Ratio of the reduced angle.

165. As examples of the reduction of the angle we have:

$$\sin 700^\circ = \sin(360^\circ + 340^\circ) = \sin 340^\circ = \sin(180^\circ + 160^\circ)$$
$$= -\sin 160^\circ = -\sin 20^\circ;$$

$$\cos(-800^\circ) = \cos 800^\circ = \cos(720^\circ + 80^\circ) = \cos 80^\circ;$$

$$\tan 500^\circ = \tan(360^\circ + 140^\circ) = \tan 140^\circ = -\tan 40^\circ;$$

$$\cot 460^\circ = \cot(360^\circ + 100^\circ) = \cot 100^\circ = -\cot 80^\circ.$$

$$\sec 930^\circ = \sec(720^\circ + 210^\circ) = \sec 210^\circ = \sec(180^\circ + 30^\circ)$$
$$= -\sec 30^\circ;$$

$$\operatorname{cosec}(-600^\circ) = -\operatorname{cosec} 600^\circ = -\operatorname{cosec}(360^\circ + 240^\circ)$$
$$= -\operatorname{cosec} 240^\circ = -\operatorname{cosec}(180^\circ + 60^\circ) = \operatorname{cosec} 60^\circ.$$

EXAMPLES. XVII.

Find the values of the following twelve Trigonometrical Ratios:

1. $\sin 225^\circ$.
2. $\sin 810^\circ$.
3. $\sin(-240^\circ)$.
4. $\cos 210^\circ$.
5. $\cos 540^\circ$.
6. $\cos(-300^\circ)$.
7. $\tan 195^\circ$.
8. $\tan 345^\circ$.
9. $\tan(-120^\circ)$
10. $\cot 420^\circ$.
11. $\cot 510^\circ$.
12. $\cot(-315^\circ)$.

13. With the notation of Chapter XIV. shew that the area of a triangle is equal to

$$Rr(\sin A + \sin B + \sin C).$$

14. Shew also that the area is equal to

$$\tfrac{1}{2}R^2(\sin 2A + \sin 2B + \sin 2C).$$

15. From the bottom of a station in a horizontal plane the altitude of the summit of a mountain is found to be α, and on retiring c feet from the station its top is seen to be in a straight line with the top of the mountain: shew that if h be the height of the station the height of the mountain is $\dfrac{ch}{c - h\cot\alpha}$ feet.

16. If CD subtend an angle α at each of the stations A and B, which are distant h apart, and the sum of the two angles ABD and BAC be σ, shew that the distance between C and D is either

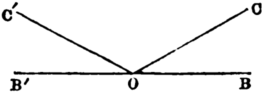
$$\frac{h\sin\alpha}{\sin(\sigma-\alpha)} \text{ or } \frac{h\sin\alpha}{\sin(\sigma+\alpha)}.$$

XVIII. *Angles with given Trigonometrical Ratios.*

166. We have shewn in Art. 17 that corresponding to a given angle there is only one value for an assigned Trigonometrical Ratio. But corresponding to a given value of an assigned Trigonometrical Ratio there is an unlimited number of angles, as we see from Chapter xv.

We shall now investigate expressions which include all the angles having a given value of an assigned Trigonometrical Ratio.

167. *To find an expression for all the angles which have a given sine.*



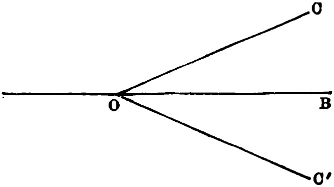
Let BOC be the least positive angle which has the given sine; denote this angle by A. Produce BO to any point B' and make the angle $B'OC'=BOC$; then $BOC' = 180^\circ - A$.

Now it is obvious from the figure that the only *positive* angles which have the same sine as A are $180^\circ - A$, and the angles formed by adding any multiple of four right angles to A or to $180^\circ - A$; that is, angles included in the expressions $n360^\circ + A$ and $n360^\circ + 180^\circ - A$, where n is zero or any positive integer. Also the only *negative* angles which have the same sine as A are $-(180^\circ + A)$ and $-(360^\circ - A)$, and the angles formed by adding to these any multiple of four right angles taken negatively; that is, angles included in the expressions $n360^\circ - (180^\circ + A)$, and $n360^\circ - (360^\circ - A)$, where n is zero or any negative integer.

All the angles which have been indicated will be found on trial to be included in the expression $n180^\circ + (-1)^n A$, where n is zero or any integer positive or negative. Also all the angles included in this expression will be found among the angles which have been indicated.

Thus this expression includes all the angles which have the same sine as A; and all the angles which it includes have the same sine as A. This expression also applies for all the angles which have the same *cosecant* as A.

168. *To find an expression for all the angles which have a given cosine.*



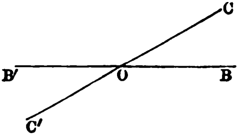
Let BOC be the least positive angle which has the given cosine; denote this angle by A. Make the angle $BOC' = BOC$.

Now it is obvious from the figure that the only *positive* angles which have the same cosine as A are $360^0 - A$, and the angles formed by adding any multiple of four right angles to A or to $360^0 - A$; that is, angles included in the expressions $n360^0 + A$ and $n360^0 + 360^0 - A$, where n is zero or any positive integer. Also the only *negative* angles which have the same cosine as A are $-A$ and $-(360^0 - A)$, and the angles formed by adding to these any multiple of four right angles taken negatively; that is, angles included in the expressions $n360^0 - A$, and $n360^0 - (360^0 - A)$, where n is zero or any negative integer.

All the angles which have been indicated will be found on trial to be included in the expression $n360^0 \pm A$, where n is zero or any integer positive or negative. Also all the angles included in this expression will be found among the angles which have been indicated.

Thus this expression includes all the angles which have the same cosine as A; and all the angles which it includes have the same cosine as A. This expression also applies for all the angles which have the same *secant* as A.

169. *To find an expression for all the angles which have a given tangent.*



Let BOC be the least positive angle which has the given tangent; denote this angle by A. Produce BO to any point B', and CO to any point C'.

Now it is obvious from the figure that the only *positive* angles which have the same tangent as A are $180^\circ + A$, and the angles formed by adding any multiple of four right angles to A or to $180^\circ + A$; that is angles included in the expressions $n360^\circ + A$ and $n360^\circ + 180^\circ + A$, where n is zero or any positive integer. Also the only *negative* angles which have the same tangent as A are $-(180^\circ - A)$ and $-(360^\circ - A)$, and the angles formed by adding to these any multiple of four right angles taken negatively; that is, angles included in the expressions $n360^\circ - (180^\circ - A)$ and $n360^\circ - (360^\circ - A)$, where n is zero or any negative integer.

All the angles which have been indicated will be found on trial to be included in the expression $n180^\circ + A$, where n is zero or any integer positive or negative. Also all the angles included in this expression will be found among the angles which have been indicated.

Thus this expression includes all the angles which have the same tangent as A; and all the angles which it includes have the same tangent as A. This expression also applies for all the angles which have the same *cotangent* as A.

EXAMPLES. XVIII.

Give the general solutions of the following eight equations:

1. $\sin A = \frac{1}{2}$. 2. $\cos A = \frac{1}{2}$. 3. $\tan A = \sqrt{3}$.

4. $\operatorname{cosec} A = 1$. 5. $\sec A = -1$. 6. $\cot A = 2 - \sqrt{3}$.

7. $\sin A + \operatorname{cosec} A = 2$. 8. $\sin 2A = \cos 3A$.

9. Shew that the length of the straight line drawn to bisect the angle A of a triangle and terminated by the opposite side is $\dfrac{2bc \cos \frac{1}{2}A}{b+c}$.

10. The angle C of a triangle is a right angle. Shew that the radius of the inscribed circle is equal to

$$\frac{1}{2}\{a + b - \sqrt{(a^2 + b^2)}\}.$$

11. Shew that the radius of a circle which passes through the vertex A of a triangle, and touches the side BC at its middle point is

$$\frac{2(b^2 + c^2) - a^2}{8b \sin C}.$$

12. P, Q, and R are points in the sides BC, CA, and AB respectively of a triangle, such that

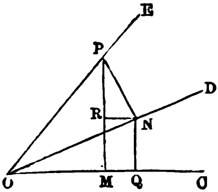
$$\frac{BP}{BC} = \frac{CQ}{CA} = \frac{AR}{AB} = x:$$

shew that $PQ^2 + QR^2 + RP^2 = (a^2 + b^2 + c^2)(1 - 3x + 3x^2)$.

XIX. *Trigonometrical Ratios of two angles.*

170. The object of the present Chapter is to express the Trigonometrical Ratios of the sum or difference of two angles in terms of the Trigonometrical Ratios of the angles themselves.

171. *To express the sine of the sum of two angles in terms of the sines and cosines of the angles themselves.*



Let the angle COD be denoted by A, and the angle DOE by B; then the angle COE will be denoted by $A+B$.

In OE take any point P, draw PM perpendicular to OC, and PN perpendicular to OD; draw NR perpendicular to PM, and NQ perpendicular to OC.

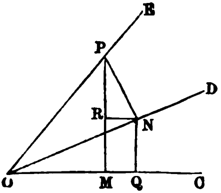
Then the angle NPR is the complement of PNR, and is therefore equal to RNO, which is equal to NOQ or A.

$$\text{Now } \sin(A+B)=\frac{PM}{OP}=\frac{RM+PR}{OP}=\frac{NQ}{OP}+\frac{PR}{OP}$$

$$=\frac{NQ}{ON}\cdot\frac{ON}{OP}+\frac{PR}{PN}\cdot\frac{PN}{OP}$$

$$=\sin A\cos B+\cos A\sin B.$$

172. *To express the cosine of the sum of two angles in terms of the sines and cosines of the angles themselves.*



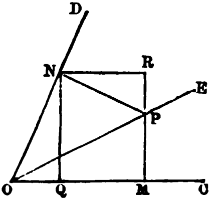
The same construction being made as in the preceding Article, we have

$$\cos(A+B)=\frac{OM}{OP}=\frac{OQ-QM}{OP}=\frac{OQ}{OP}-\frac{NR}{OP}$$

$$=\frac{OQ}{ON}\cdot\frac{ON}{OP}-\frac{NR}{NP}\cdot\frac{NP}{OP}$$

$$=\cos A\cos B-\sin A\sin B.$$

173. *To express the sine of the difference of two angles in terms of the sines and cosines of the angles themselves.*



Let the angle *COD* be denoted by A, and the angle *DOE* by B; then the angle *COE* will be denoted by $A-B$.

In *OE* take any point *P*, draw *PM* perpendicular to *OC*, and *PN* perpendicular to *OD*; draw *NR* perpendicular to *MP* produced, and *NQ* perpendicular to *OC*.

Then the angle NPR is the complement of PNR, and is therefore equal to DNR, which is equal to NOQ or A.

$$\text{Now } \sin(A-B)=\frac{PM}{OP}=\frac{RM-RP}{OP}=\frac{NQ}{OP}-\frac{RP}{OP}$$

$$=\frac{NQ}{ON}\cdot\frac{ON}{OP}-\frac{RP}{PN}\cdot\frac{PN}{OP}$$

$$=\sin A\cos B-\cos A\sin B.$$

174. *To express the cosine of the difference of two angles in terms of the sines and cosines of the angles themselves.*

The same construction being made as in the preceding Article we have

$$\cos(A-B)=\frac{OM}{OP}=\frac{OQ+QM}{OP}=\frac{OQ}{OP}+\frac{NR}{OP}$$

$$=\frac{OQ}{ON}\cdot\frac{ON}{OP}+\frac{NR}{PN}\cdot\frac{PN}{OP}$$

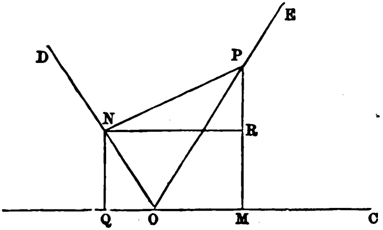
$$=\cos A\cos B+\sin A\sin B.$$

175. To assist the student in remembering the preceding demonstrations, we may observe that the point P is taken in the straight line which *bounds the compound angle we are considering;* thus in demonstrating the formulæ for $\sin(A+B)$ and $\cos(A+B)$ the point P is taken in the straight line which bounds the angle $A+B$, and in demonstrating the formulæ for $\sin(A-B)$ and $\cos(A-B)$ the point P is taken in the straight line which bounds the angle $A-B$.

176. The formulæ established in Arts. 171...174 are true whatever may be the size of the angles A and B; the student may exercise himself by going through the construction and demonstration in various cases; it will be found that the only variety which occurs in the construction consists in the circumstance that the perpendiculars in

some cases fall on certain straight lines and in other cases fall on those straight lines *produced.*

177. We will for example demonstrate the formulæ of Arts. 173, 174, for the case in which the angle A is greater than a right angle, while B is less than a right angle, and $A-B$ is less than a right angle.



Let the angle COD be denoted by A, and the angle DOE by B; then the angle COE will be denoted by $A-B$.

In OE take any point P, draw PM perpendicular to OC and PN perpendicular to OD; draw NR perpendicular to MP and NQ perpendicular to CO produced.

Then the angle NPR is the complement of PNR, and is therefore equal to RNO, which is equal to $180^\circ - A$. Then

$$\sin(A-B) = \frac{PM}{OP} = \frac{RM+PR}{OP} = \frac{NQ}{OP} + \frac{PR}{OP}$$

$$= \frac{NQ}{ON} \cdot \frac{ON}{OP} + \frac{PR}{PN} \cdot \frac{PN}{OP}$$

$$= \sin(180^\circ - A)\cos B + \cos(180^\circ - A)\sin B$$

$$= \sin A \cos B - \cos A \sin B, \text{ by Art. 95.}$$

And $\cos(A-B)$

$$=\frac{OM}{OP}=\frac{MQ-OQ}{OP}=\frac{NR}{OP}-\frac{OQ}{OP}$$

$$=\frac{NR}{PN}\cdot\frac{PN}{OP}-\frac{OQ}{ON}\cdot\frac{ON}{OP}$$

$$=\sin(180^\circ-A)\sin B-\cos(180^\circ-A)\cos B$$

$$=\sin A\sin B+\cos A\cos B, \text{ by Art. 95.}$$

178. By examining in this manner the various cases which can occur, the student may convince himself that the formulæ of Arts. 171...174 are universally true. Another mode of demonstration will be found in the larger work on Trigonometry, Art. 80.

We now proceed to express the tangent and cotangent of a compound angle in terms of the tangents and cotangents of the single angles.

179. $\operatorname{Tan}(A+B)=\dfrac{\sin(A+B)}{\cos(A+B)}$

$$=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B};$$

divide both numerator and denominator of the last expression by $\cos A \cos B$; thus we obtain

$$\frac{\dfrac{\sin A}{\cos A}+\dfrac{\sin B}{\cos B}}{1-\dfrac{\sin A}{\cos A}\,\dfrac{\sin B}{\cos B}};$$

therefore $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}.$

180. Tan $(A-B)$

$$=\frac{\sin(A-B)}{\cos(A-B)}=\frac{\sin A\cos B-\cos A\sin B}{\cos A\cos B+\sin A\sin B}$$

$$=\frac{\dfrac{\sin A}{\cos A}-\dfrac{\sin B}{\cos B}}{1+\dfrac{\sin A}{\cos A}\dfrac{\sin B}{\cos B}}=\frac{\tan A-\tan B}{1+\tan A\tan B}.$$

181. Cot $(A+B)$

$$=\frac{\cos(A+B)}{\sin(A+B)}=\frac{\cos A\cos B-\sin A\sin B}{\sin A\cos B+\cos A\sin B}$$

$$=\frac{\dfrac{\cos A}{\sin A}\dfrac{\cos B}{\sin B}-1}{\dfrac{\cos A}{\sin A}+\dfrac{\cos B}{\sin B}}=\frac{\cot A\cot B-1}{\cot A+\cot B}.$$

182. Cot $(A-B)$

$$=\frac{\cos(A-B)}{\sin(A-B)}=\frac{\cos A\cos B+\sin A\sin B}{\sin A\cos B-\cos A\sin B}$$

$$=\frac{\dfrac{\cos A}{\sin A}\dfrac{\cos B}{\sin B}+1}{\dfrac{\cos B}{\sin B}-\dfrac{\cos A}{\sin A}}=\frac{\cot A\cot B+1}{\cot B-\cot A}.$$

183. We may give other forms to these expressions.

For example,

$$\cot(A+B)=\frac{1}{\tan(A+B)}=\frac{1-\tan A\tan B}{\tan A+\tan B},$$

$$\cot(A-B)=\frac{1}{\tan(A-B)}=\frac{1+\tan A\tan B}{\tan A-\tan B}.$$

184. Some important formulæ are deducible from those already given by supposing $B=A$.

Write $B=A$ in Art. 171; thus

$$\sin 2A=2\sin A\cos A.$$

Write $B=A$ in Art. 172; thus

$$\begin{aligned}\cos 2A&=\cos^2 A-\sin^2 A\\&=1-2\sin^2 A\\\text{or }&=2\cos^2 A-1.\end{aligned}$$

Thus $\quad 1+\cos 2A=2\cos^2 A,$

$1-\cos 2A=2\sin^2 A,$

and $\quad \dfrac{1-\cos 2A}{1+\cos 2A}=\tan^2 A$; also $\dfrac{\sin 2A}{1+\cos 2A}=\tan A.$

And since $\sin^2 A+\cos^2 A=1$ we have

$$\sin 2A=\frac{2\sin A\cos A}{\cos^2 A+\sin^2 A};$$

divide both numerator and denominator of the last expression by $\cos^2 A$; thus

$$\sin 2A=\frac{2\tan A}{1+\tan^2 A}.$$

Similarly $\cos 2A=\dfrac{\cos^2 A-\sin^2 A}{\cos^2 A+\sin^2 A}=\dfrac{1-\tan^2 A}{1+\tan^2 A}.$

In Art. 179 put $B=A$; thus

$$\tan 2A=\frac{2\tan A}{1-\tan^2 A},$$

and this result may also be deduced from the values of $\sin 2A$ and $\cos 2A$ just given.

In Art. 181, put $B=A$; thus

$$\cot 2A=\frac{\cot^2 A-1}{2\cot A}.$$

EXAMPLES. XIX.

1. If $\tan A = \frac{4}{3}$ and $\tan B = 7$, find $\tan (A + B)$.

2. If $\tan A = \frac{1}{\sqrt{3}}$ and $\tan B = \frac{1}{\sqrt{15}}$, find $\sin (A + B)$.

3. If $\tan B = \frac{2 \sin A \sin C}{\sin (A + C)}$, shew that $\tan A$, $\tan B$, and $\tan C$ are in Harmonical Progression.

4. If $\tan A = a$ and $\tan B = b$, find $\cos 2 (A + B)$ and $\sin 2 (A + B)$.

5. If $\cos (A - B) = n \sin (A + B)$, shew that

$$\tan (45^\circ + A) = \frac{n+1}{n-1} \tan (45^\circ - B).$$

6. If $\sin A = \frac{1}{\sqrt{3}}$ and $\sin B = \frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{3}}$, find $\cos (A + B)$.

7. Given $\tan 2A = \frac{24}{7}$, find $\sin A$.

8. If $\tan 2A = 2 \tan B$ and $\tan C = \tan^3 A$, shew that $\tan (B - C) = \tan A$.

9. If $\cos A = \frac{p + qc}{q + pc}$ and $\cos B = \frac{p - qc}{q - pc}$, find $\cos (A + B)$.

10. If $\tan A = \tan^3 B$, then $2\tan 2A = \sin 2B \tan 2B$.

11. Given $\sin A - \cos A = \frac{1}{\sqrt{2}} \cos 1^\circ - \frac{\sqrt{3}}{\sqrt{2}} \sin 1^\circ$, find the number of degrees in the smallest possible value of A.

12. If $\tan B + \cot B = 2 \sec 2A$, shew that one value of $A + B$ is 45°.

13. Shew that $\sec 2A - \tan 2A = \tan(45^\circ - A)$.

14. If $\cot B - 2\cot 2B = \sec 2A - \tan 2A$, shew that one value of $A + B$ is 45°.

15. If $p \sin A = q \sin B$ and $r \cos A = s \cos B$, then

$$\sin(A \pm B) = \frac{\sqrt{(p^2 - q^2)}\sqrt{(s^2 - r^2)}}{qr \mp ps}, \quad \cos(A \pm B) = \frac{qs \mp pr}{qr \mp ps}.$$

Demonstrate the following ten identities:

16. $\dfrac{\sin 2A}{1 + \cos 2A} \cdot \dfrac{\cos A}{1 + \cos A} = \tan \dfrac{A}{2}$.

17. $\sin 8A = 8 \sin A \cos A \cos 2A \cos 4A$.

18. $(\sin A - \sin B)^2 + (\cos A - \cos B)^2 = 2 \operatorname{vers}(A - B)$.

19. $2 \operatorname{cosec} 4A + 2 \cot 4A = \cot A - \tan A$.

20. $2 \sin 2A - \sin 4A = 4 \sin 2A \sin^2 A$.

21. $\cot^2 A - \tan^2 A = 4 \cot 2A \operatorname{cosec} 2A$.

22. $2 - 2 \tan A \cot 2A = \sec^2 A$.

23. $\sec^2(A + 45^\circ) - \sec^2(A - 45^\circ) = 4 \tan 2A \sec 2A$.

24. $\tan A + \cot 2A = \operatorname{cosec} 2A$.

25. $\tan(45^\circ + A) - \tan(45^\circ - A) = 2 \tan 2A$.

XX. *Trigonometrical Transformations.*

185. From the formulæ of the preceding Chapter others may be derived; and by the aid of all the results we can change Trigonometrical expressions into various forms. The use of such transformations becomes apparent as the student advances in Mathematics, and even at present he will find valuable exercise in them.

186. The product of $\sin(A+B)$ and $\sin(A-B)$ takes a remarkable form.

$$\begin{aligned}
&\operatorname{Sin}(A+B)\sin(A-B)\\
&=(\sin A\cos B+\cos A\sin B)(\sin A\cos B-\cos A\sin B)\\
&=\sin^2 A\cos^2 B-\cos^2 A\sin^2 B\\
&=\sin^2 A(1-\sin^2 B)-(1-\sin^2 A)\sin^2 B\\
&=\sin^2 A-\sin^2 B.
\end{aligned}$$

This may also be put in the form $\cos^2 B-\cos^2 A$.

187. Also

$$\begin{aligned}
&\cos(A+B)\cos(A-B)\\
&=(\cos A\cos B-\sin A\sin B)(\cos A\cos B+\sin A\sin B)\\
&=\cos^2 A\cos^2 B-\sin^2 A\sin^2 B\\
&=\cos^2 A(1-\sin^2 B)-(1-\cos^2 A)\sin^2 B\\
&=\cos^2 A-\sin^2 B\\
&=\cos^2 B-\sin^2 A.
\end{aligned}$$

188. From the four fundamental formulæ of Chapter XIX. we have

$$\sin(A+B)+\sin(A-B)=2\sin A\cos B,$$

$$\sin(A+B)-\sin(A-B)=2\cos A\sin B,$$

$$\cos(A+B)+\cos(A-B)=2\cos A\cos B,$$

$$\cos(A-B)-\cos(A+B)=2\sin A\sin B.$$

Let $A+B=C$, and $A-B=D$; therefore

$$A=\frac{C+D}{2},\quad B=\frac{C-D}{2}: \text{ thus}$$

$$\sin C+\sin D=2\sin\frac{C+D}{2}\cos\frac{C-D}{2},$$

$$\sin C-\sin D=2\cos\frac{C+D}{2}\sin\frac{C-D}{2},$$

$$\cos C+\cos D=2\cos\frac{C+D}{2}\cos\frac{C-D}{2},$$

$$\cos D-\cos C=2\sin\frac{C+D}{2}\sin\frac{C-D}{2}.$$

The last four formulæ are useful in converting a sum or difference into the form of a product of factors. Thus, for example, by the first of the last four formulæ the sum of two sines is converted into twice the product of a sine and cosine.

The first four formulæ are useful in converting a product into the form of a sum or difference. Thus, for example, by the last of the first four formulæ we see that the product of two sines is equal to half the cosine of the difference of the two angles diminished by half the cosine of their sum.

189. $$\frac{\sin A+\sin B}{\sin A-\sin B}=\frac{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}{2\cos\frac{A+B}{2}\sin\frac{A-B}{2}} \text{ (Art. 188)}$$

$$=\tan\frac{A+B}{2}\cot\frac{A-B}{2}$$

$$=\frac{\tan\frac{A+B}{2}}{\tan\frac{A-B}{2}}.$$

190. $$\frac{\cos A+\cos B}{\cos B-\cos A}=\frac{2\cos\frac{A+B}{2}\cos\frac{A-B}{2}}{2\sin\frac{A+B}{2}\sin\frac{A-B}{2}} \text{ (Art. 188)}$$

$$=\cot\frac{A+B}{2}\cot\frac{A-B}{2}.$$

191. $$\tan A+\tan B=\frac{\sin A}{\cos A}+\frac{\sin B}{\cos B}$$

$$=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B}$$

$$=\frac{\sin(A+B)}{\cos A\cos B}.$$

Similarly $$\tan A-\tan B=\frac{\sin(A-B)}{\cos A\cos B}.$$

192. $$\tan A+\cot A=\frac{\sin A}{\cos A}+\frac{\cos A}{\sin A}=\frac{\sin^2 A+\cos^2 A}{\sin A\cos A}$$

$$=\frac{1}{\sin A\cos A}=\frac{2}{2\sin A\cos A}=\frac{2}{\sin 2A}.$$

193. $$\cot A - \tan A = \frac{\cos A}{\sin A} - \frac{\sin A}{\cos A}$$
$$= \frac{\cos^2 A - \sin^2 A}{\sin A \cos A} = \frac{\cos 2A}{\sin A \cos A}$$
$$= \frac{2 \cos 2A}{2 \sin A \cos A} = \frac{2 \cos 2A}{\sin 2A} = 2 \cot 2A.$$

194. By repeated applications of the formulæ in Chapter XIX, we can obtain expressions for the Trigonometrical Ratios of a composite angle made up of any number of simple angles connected by the signs + and –. For example,

$$\sin (A+B+C) = \sin (A+B) \cos C + \cos (A+B) \sin C$$
$$= \sin A \cos B \cos C + \sin B \cos C \cos A$$
$$+ \sin C \cos A \cos B - \sin A \sin B \sin C.$$

$$\cos (A+B+C) = \cos (A+B) \cos C - \sin (A+B) \sin C;$$
$$= \cos A \cos B \cos C - \cos C \sin A \sin B$$
$$- \cos B \sin A \sin C - \cos A \sin B \sin C.$$

$$\tan (A+B+C) = \frac{\sin (A+B+C)}{\cos (A+B+C)}:$$

substitute the expressions just found for $\sin (A+B+C)$ and $\cos (A+B+C)$; and then divide both numerator and denominator by $\cos A \cos B \cos C$; thus we obtain

$$\tan (A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C - \tan C \tan A - \tan A \tan B}.$$

195. The particular case of the formulæ in the preceding Article in which $C = B = A$ should be noticed. Thus we obtain

$$\sin 3A = 3 \sin A \cos^2 A - \sin^3 A$$
$$= 3 \sin A (1 - \sin^2 A) - \sin^3 A$$
$$= 3 \sin A - 4 \sin^3 A;$$

$$\begin{aligned}\cos 3A &= \cos^3 A - 3\cos A\sin^2 A\\ &= \cos^3 A - 3\cos A\,(1-\cos^2 A)\\ &= 4\cos^3 A - 3\cos A\,;\end{aligned}$$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}.$$

196. When angles are connected by a relation we often find some simple relation connecting some of their Trigonometrical Ratios. We will take for example the case in which there are three angles, the sum of which is equal to 180^0; this relation holds for the angles of a triangle.

If $A+B+C=180^0$, then will

$$\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

For $$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}\ \text{(Art. 188)}$$

$$\begin{aligned}&= 2\sin\left(90^0 - \frac{C}{2}\right)\cos\frac{A-B}{2}\\ &= 2\cos\frac{C}{2}\cos\frac{A-B}{2}\,;\end{aligned}$$

$$\begin{aligned}\sin C &= 2\sin\frac{C}{2}\cos\frac{C}{2}\\ &= 2\cos\frac{C}{2}\cos\left(90^0 - \frac{C}{2}\right)\\ &= 2\cos\frac{C}{2}\cos\frac{A+B}{2}\,;\end{aligned}$$

therefore $\sin A + \sin B + \sin C$

$$= 2\cos\frac{C}{2}\left\{\cos\frac{A-B}{2} + \cos\frac{A+B}{2}\right\}$$

$$= 2\cos\frac{C}{2}\, 2\cos\frac{A}{2}\cos\frac{B}{2} \text{ (Art. 188)}$$

$$= 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

Again, if $A+B+C=180^0$, then will

$$\cot\frac{B}{2} + \cot\frac{C}{2} = \frac{\cos\frac{A}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}}.$$

For $$\cot\frac{B}{2} + \cot\frac{C}{2} = \frac{\cos\frac{B}{2}}{\sin\frac{B}{2}} + \frac{\cos\frac{C}{2}}{\sin\frac{C}{2}}$$

$$= \frac{\cos\frac{B}{2}\sin\frac{C}{2} + \sin\frac{B}{2}\cos\frac{C}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$= \frac{\sin\frac{B+C}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}} \text{ (Art. 171)}$$

$$= \frac{\sin\left(90^0 - \frac{A}{2}\right)}{\sin\frac{B}{2}\sin\frac{C}{2}} = \frac{\cos\frac{A}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}}.$$

Similarly if $A+B+C=180^\circ$, then

$$\tan\frac{B}{2}+\tan\frac{C}{2}=\frac{\cos\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}.$$

197. The solution of equations involving Trigonometrical Ratios is facilitated by transformations of the kind given in the present Chapter. For example; required to find the value of the angle A from the equation

$$\sin 7A-\sin A=\sin 3A.$$

By Art. 188,

$$\sin 7A-\sin A=2\sin 3A\cos 4A;$$

thus we have the equation

$$2\sin 3A\cos 4A=\sin 3A;$$

therefore either

$$\sin 3A=0,$$

or

$$\cos 4A=\frac{1}{2}.$$

The former gives as the simplest solution $3A=0$, and for the general solution $3A=n\,180^\circ$.

The latter gives as the simplest solution $4A=60^\circ$, and for the general solution $4A=n\,360^\circ\pm 60^\circ$.

See Chapter XVIII.

EXAMPLES. XX.

Demonstrate the following thirty identities :

1. $\dfrac{\sin 3A+\sin A}{\cos 3A+\cos A}=\tan 2A.$

2. $\dfrac{\cos A-\cos 3A}{\sin A+\sin 3A}=\tan A.$

3. $\dfrac{\sin 5A-\sin 3A}{\cos 3A-\cos 5A}=\cot 4A.$

4. $\dfrac{\sin^2 A-\sin^2 B}{\sin A\cos A-\sin B\cos B}=\tan(A+B).$

5. $\cos^2 2A-\cos^2 3A=\sin A\sin 5A.$

6. $\sin A\sin(A+2B)-\sin B\sin(2A+B)=\sin^2 A-\sin^2 B.$

7. $\dfrac{\sin A+\sin(A+B)+\sin(A+2B)}{\cos A+\cos(A+B)+\cos(A+2B)}=\tan(A+B).$

8. $2\cos(n-1)A\cos A-\cos(n-2)A=\cos nA.$

9. $$2\sin\left(\frac{A+B}{2}-45^0\right)\cos\left(\frac{A-B}{2}+45^0\right)$$
$$=\sin A-\cos B.$$

10. $(1+\cot A+\operatorname{cosec} A)(1+\cot A-\operatorname{cosec} A)$
$$=\cot\frac{A}{2}-\tan\frac{A}{2}.$$

11. $\dfrac{\tan 3A}{\tan A}=\dfrac{2\cos 2A+1}{2\cos 2A-1}.$

12. $\sin 3A - \cos 3A = (\sin A + \cos A)(4 \sin A \cos A - 1)$.

13. $\sin B + \dfrac{\sin (A - B) - \sin B}{1 + \cos A} = \tan \dfrac{A}{2} \cos B$.

14. $2 \sec (A + B)(\cos^2 A - \sin^2 B)$
$= (\sin 2A + \sin 2B) \operatorname{cosec} (A + B)$.

15. $2 \sin \dfrac{3A}{2} \cos \dfrac{3A}{2} \cos 2A - 2 \sin A \cos A \cos 3A$
$= \sin A$.

16. $2 (\cos^8 A - \sin^8 A) = \cos 2A (1 + \cos^2 2A)$.

17. $\sin (3A + B) \sin (3A - B) - \sin (A + B) \sin (A - B)$
$= \sin 4A \sin 2A$.

18. $\{\cos (A + B) + \sin (A - B)\}\{\sin (A + B) + \cos (A - B)\}$
$= \cos 2B (1 + \sin 2A)$.

19. $\dfrac{3 \sin A - \sin 3A}{3 \cos A + \cos 3A} = \left(\dfrac{\sec 2A - 1}{\sec 2A + 1}\right)^{\frac{3}{2}}$.

20. $(\cos A - \sin A)(\cos 2A - \sin 2A) + \sin 3A = \cos A$.

21. $(3 \sin A - 4 \sin^3 A)^2 + (4 \cos^3 A - 3 \cos A)^2 = 1$.

22. $1 - \cos 3A = (1 - \cos A)(1 + 2 \cos A)^2$.

23. $\tan \dfrac{A}{2} + 2 \sin^2 \dfrac{A}{2} \cot A = \sin A$.

24. $\cos A - \sin A \tan \dfrac{A}{2} = \cos 2A + \sin 2A \tan \dfrac{A}{2}$.

25. $\dfrac{2 \cos 2A}{\cos A + \sin A} + \dfrac{2 \sin 2A}{\cos A - \sin A} = \dfrac{\sqrt{2}}{\cos (A + 45^\circ)}$.

26. $\cos(A+B-C)+\cos(A-B+C)+\cos(B+C-A)$
$$+\cos(A+B+C)=4\cos A\cos B\cos C.$$

27. $\cos x+\cos y+\cos z+\cos(x+y+z)$
$$=4\cos\frac{x+y}{2}\cos\frac{y+z}{2}\cos\frac{z+x}{2}.$$

28. $\sin(A+B-C)+\sin(A+C-B)+\sin(B+C-A)$
$$-\sin(A+B+C)=4\sin A\sin B\sin C.$$

29. $\sin x+\sin y+\sin z-\sin(x+y+z)$
$$=4\sin\frac{x+y}{2}\sin\frac{y+z}{2}\sin\frac{z+x}{2}.$$

30. $\sin(A+B+C)+\sin(B+C-A)+\sin(A+C-B)$
$$-\sin(A+B-C)=4\sin C\cos A\cos B.$$

If $A+B+C=180^0$ demonstrate the following five relations:

31. $\sin 2A+\sin 2B-\sin 2C=4\sin C\cos A\cos B.$

32. $\sin\frac{A}{2}\cos\frac{B-C}{2}+\sin\frac{B}{2}\cos\frac{C-A}{2}+\sin\frac{C}{2}\cos\frac{A-B}{2}$
$$=\cos A+\cos B+\cos C.$$

33. $\sin^2 A+\sin^2 B+\sin^2 C$
$$=2\sin B\sin C\cos A+2\sin A\sin C\cos B+2\sin A\sin B\cos C.$$

34. $\dfrac{1-\cos A+\cos B+\cos C}{1-\cos C+\cos A+\cos B}=\tan\frac{A}{2}\cot\frac{C}{2}.$

35. $\sin 2B(1+2\cos 2C)+\sin 2C(1+2\cos 2A)$
$$+\sin 2A(1+2\cos 2B)$$
$$=4\sin(C-B)\sin(A-C)\sin(B-A).$$

Solve the following five equations:

36. $\sin x + \cos 2x = 1.$

37. $\tan x = \tan 4x.$

38. $\sin 9x + \sin 5x + 2\sin^2 x = 1.$

39. $\sin 5x \cos 3x = \sin 9x \cos 7x.$

40. $\cos ax \cos bx = \cos (a+c) x \cos (b+c) x.$

41. If $\tan A = 2$, $\tan B = 3$, $\tan C = 2 + \sqrt{3}$, find

$$\tan (A + B - C).$$

42. If $\tan A = \frac{1}{4}$, $\tan B = \frac{1}{5}$, $\tan C = \frac{1}{6}$, $\tan D = \frac{1}{8}$, $\tan E = \frac{3}{56}$, shew that $\tan (A + B + C + D + E) = 1.$

43. If $2S = A + B + C$, shew that

$$\cos \frac{S}{2} \cos \frac{S-A}{2} \cos \frac{S-B}{2} \cos \frac{S-C}{2}$$

$$+ \sin \frac{S}{2} \sin \frac{S-A}{2} \sin \frac{S-B}{2} \sin \frac{S-C}{2}$$

$$= \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}.$$

44. If $2S = A + B + C$, shew that

$$\cos^2 S + \cos^2 (S-A) + \cos^2 (S-B) + \cos^2 (S-C)$$

$$= 2 + 2\cos A \cos B \cos C.$$

45. If $\text{vers}\, A = x$, $\text{vers}\, B = bx$, $\text{vers}\, C = 1 - b$, and $A + B = C$, find x.

Demonstrate the following ten identities:

46. $\cos A+\cos B+\cos(A+C)+\cos(B+C)$

$$=4\cos\frac{A+B+C}{2}\cos\frac{C}{2}\cos\frac{B-A}{2}.$$

47. $\tan\frac{A}{2}-\tan\frac{A}{3}-\tan\frac{A}{6}=\tan\frac{A}{2}\tan\frac{A}{3}\tan\frac{A}{6}.$

48. $\sin^2(B-A)+\sin^2(B-A+C)$

$$-2\sin(B-A)\sin(B-A+C)\cos C=\sin^2 C.$$

49. $4\cos 3A\sin^3(60^\circ-A)-4\sin 3A\cos^3(60^\circ-A)$

$$=3\sin(60^\circ-4A).$$

50. $\cos A+\cos(72^\circ-A)+\cos(72^\circ+A)+\cos(144^\circ-A)$ $+\cos(144^\circ+A)=0.$

51. $1+\cos(A-B)+\cos(B-C)+\cos(C-A)$

$$=4\cos\frac{A-B}{2}\cos\frac{B-C}{2}\cos\frac{C-A}{2}.$$

52. $\sin(A-B)+\sin(B-C)+\sin(C-A)$

$$=4\sin\frac{B-A}{2}\sin\frac{C-B}{2}\sin\frac{A-C}{2}.$$

53. $\sin 3(A-15^\circ)$

$$=4\cos(A-45^\circ)\cos(A+15^\circ)\sin(A-15^\circ).$$

54. $$\frac{\sin(2A-B-C)}{\sin(A-B)\sin(A-C)}+\frac{\sin(2B-A-C)}{\sin(B-A)\sin(B-C)}+\frac{\sin(2C-A-B)}{\sin(C-A)\sin(C-B)}=0.$$

55. $$\frac{\cos(2A-B-C)}{\sin(A-B)\sin(A-C)}+\frac{\cos(2B-A-C)}{\sin(B-A)\sin(B-C)}+\frac{\cos(2C-A-B)}{\sin(C-A)\sin(C-B)}=-4.$$

XXI. *Division of Angles.*

198. By Art. 184, we have

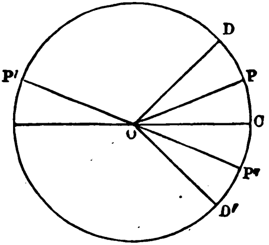
$$\cos A = 2\cos^2\frac{A}{2} - 1 = 1 - 2\sin^2\frac{A}{2};$$

hence $\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$, $\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$.

These formulæ serve to determine the sine and the cosine of half an angle, when the cosine of the angle is given. It will be seen that by reason of the double sign we have *two* values for $\sin\frac{A}{2}$ and *two* values for $\cos\frac{A}{2}$ corresponding to a given value of $\cos A$.

199. The reason why there is more than one value for $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$, corresponding to a given value of $\cos A$, is that corresponding to a given value of the cosine there is more than one value of the angle.

Thus suppose that the angle COD has its cosine equal to $\cos A$, then an angle equal to four right angles diminished by COD also has its cosine equal to $\cos A$; this



angle is denoted in the figure by the larger angle bounded by OC and OD'. If we take COD for A, then $\frac{1}{2}A$ is the angle COP, where OP is such that $COP = POD$. If we take for A the larger angle bounded by OC and OD', then $\frac{1}{2}A$ is the angle COP', where OP' is such that $COP' = P'OD'$. Also the angle measured in the negative direction between OC and OD', that is the angle COD' has its cosine equal to $\cos A$. If we take COD' for A, then $\frac{1}{2}A$ is the angle COP'', where OP'' is such that $COP'' = P''OD'$.

It is easy to see on examining the figure that COP' is the supplement of COP, and that OP' and OP'' are in the same straight line. Hence it follows that the sines of COP, COP' and COP'' are numerically equal but have not all the same sign; so also the cosines of COP, COP' and COP'' are numerically equal but have not all the same sign.

If any of the angles which we have taken for A be increased by any multiple of four right angles, we shall obtain an angle which has its cosine equal to $\cos A$; it will be found on examining the figure that the sine and the cosine of half such an angle will coincide respectively with the sine and the cosine of one of the angles which we have already taken for $\frac{1}{2}A$.

Thus we have accounted for the fact that $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$ when expressed in terms of $\cos A$ have each two values numerically equal but of opposite signs.

200. By assuming the result obtained in Art. 168 we can put the preceding explanation into a briefer form.

All the angles which are comprised in the expression $n\,360^\circ \pm A$, where n is any integer, positive or negative, have the same cosine as A. Hence we may expect that any formula which gives $\sin\frac{A}{2}$ in terms of $\cos A$ will include the sine of every angle which is comprised in the expression $\frac{1}{2}(n\,360^\circ \pm A)$, that is in the expression $n\,180^\circ \pm \frac{1}{2}A$.

Now $\sin(n\,180^\circ \pm \tfrac{1}{2}A) = \sin(\pm\tfrac{1}{2}A)$ if n be even,

and $= -\sin(\pm\tfrac{1}{2}A)$ if n be odd.

And thus, by Art. 159, we have the two values $\pm\sin\tfrac{1}{2}A$, and no more.

Similarly we may expect that any formula which gives $\cos\dfrac{A}{2}$ in terms of $\cos A$ will include the cosine of every angle comprised in the expression $n\,180^\circ \pm \tfrac{1}{2}A$.

Now $\cos(n\,180^\circ \pm \tfrac{1}{2}A) = \cos(\pm\tfrac{1}{2}A)$ if n be even,

and $= -\cos(\pm\tfrac{1}{2}A)$ if n be odd.

And thus, by Art. 159, we have the two values $\pm\cos\tfrac{1}{2}A$, and no more.

201. If in any case we actually know the value of A we know also the value of $\tfrac{1}{2}A$; and then we can settle which sign we ought to take in the formula for $\sin\tfrac{1}{2}A$, and which sign we ought to take in the formula for $\cos\tfrac{1}{2}A$. And even if we do not know the exact value of A we may know sufficient to enable us to make this selection; for example, if we know that A lies between 90° and 180°, then we know that $\tfrac{1}{2}A$ lies between 45° and 90°, and the positive sign must be taken in both the formulæ of Art. 198.

202. Remarks similar to those which have been made in the last three Articles will be found applicable also to numerous other results in Trigonometry in which the double sign occurs; for examples we may mention the remaining results of the present Chapter, or the result $\sin A = \pm\sqrt{1-\cos^2 A}$, and others of the same kind. We shall however not enlarge on this point, for we have given enough to illustrate the general principle; more applications will be found in Chapter VII. of the larger Trigonometry.

203. By Art. 184

$$\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2};$$

and $$1 = \sin^2 \frac{A}{2} + \cos^2 \frac{A}{2};$$

thus $$\left(\sin \frac{A}{2} + \cos \frac{A}{2}\right)^2 = 1 + \sin A,$$

and $$\left(\sin \frac{A}{2} - \cos \frac{A}{2}\right)^2 = 1 - \sin A;$$

therefore $$\sin \frac{A}{2} + \cos \frac{A}{2} = \pm \sqrt{\left(1 + \sin A\right)} \quad (1),$$

and $$\sin \frac{A}{2} - \cos \frac{A}{2} = \pm \sqrt{\left(1 - \sin A\right)} \quad (2).$$

Thus as soon as the proper signs are known in (1) and (2) we can by addition and subtraction find expressions for $\sin \frac{A}{2}$ and $\cos \frac{A}{2}$ in terms of $\sin A$.

Let us take the case in which A is an *acute* angle, then $\frac{A}{2}$ lies between 0^0 and 45^0; thus $\sin \frac{A}{2}$ and $\cos \frac{A}{2}$ are both positive, and $\cos \frac{A}{2}$ is greater than $\sin \frac{A}{2}$. We must therefore take the upper sign in (1) and the lower sign in (2), so that

$$\sin \frac{A}{2} + \cos \frac{A}{2} = \sqrt{(1 + \sin A)},$$

$$\sin \frac{A}{2} - \cos \frac{A}{2} = -\sqrt{(1 - \sin A)};$$

therefore $$2\sin\frac{A}{2}=\sqrt{(1+\sin A)}-\sqrt{(1-\sin A)},$$

and $$2\cos\frac{A}{2}=\sqrt{(1+\sin A)}+\sqrt{(1-\sin A)}.$$

204. As an example of the formulæ of the preceding Article we will find the sine and the cosine of an angle of 9°.

$$2\sin 9^0=\sqrt{(1+\sin 18^0)}-\sqrt{(1-\sin 18^0)},$$

$$2\cos 9^0=\sqrt{(1+\sin 18^0)}+\sqrt{(1-\sin 18^0)};$$

$$1+\sin 18^0=1+\frac{\sqrt{5}-1}{4}=\frac{3+\sqrt{5}}{4},$$

$$1-\sin 18^0=1-\frac{\sqrt{5}-1}{4}=\frac{5-\sqrt{5}}{4}.$$

Thus $$\sin 9^0=\frac{1}{4}\left\{\sqrt{(3+\sqrt{5})}-\sqrt{(5-\sqrt{5})}\right\},$$

and $$\cos 9^0=\frac{1}{4}\left\{\sqrt{(3+\sqrt{5})}+\sqrt{(5-\sqrt{5})}\right\}.$$

205. By Art. 195 we have

$$\cos A=4\cos^3\frac{A}{3}-3\cos\frac{A}{3},$$

$$\sin A=3\sin\frac{A}{3}-4\sin^3\frac{A}{3}.$$

Thus if $\cos A$ be given and we require $\cos\frac{A}{3}$ we have to solve a cubic equation; and similarly if $\sin A$ be given and we require $\sin\frac{A}{3}$ we have to solve a cubic equation.

EXAMPLES. XXI.

1. Find the cosine of $11\frac{1}{4}^\circ$.

2. If A be between 90° and 180°, shew that

$$2\sin\frac{A}{2} = \sqrt{(1+\sin A)} + \sqrt{(1-\sin A)}.$$

3. If $\sin 224^\circ = -{\cdot}69$, write down the value of $\sin 112^\circ$.

4. Shew that $\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$. Explain the double sign.

5. Shew that $\sin A$ when expressed in terms of $\sin\frac{A}{2}$ has two equal values of opposite signs; and that $\cos A$ when expressed in terms of $\cos\frac{A}{2}$ has only one value: and give a geometrical illustration.

6. Shew that

$$\cot\left(45^\circ - \frac{A}{2}\right) - \cot\left(45^\circ + \frac{A}{2}\right) = 2\tan A.$$

7. Shew that

$$\sin 5A \operatorname{cosec}^2 A \sec A - \cos 5A \sec^2 A \operatorname{cosec} A = 8\cot 2A.$$

8. Shew that

$$\frac{\cos A}{1-\cos A} - \frac{1+\cos A}{\cos A} = 2\cot 2A \cot\frac{A}{2}.$$

9. If $a\sin\theta + b\cos\theta = c = a\operatorname{cosec}\theta + b\sec\theta$,

shew that $$(a^{\frac{2}{3}} - b^{\frac{2}{3}})\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}} = c.$$

10. If

$$\sin\theta + \sin\phi = m, \quad \cos\theta + \cos\phi = n, \text{ and } \cos(\theta+\phi) = p,$$

shew that $$\frac{m^2}{n^2} = \frac{1-p}{1+p}.$$

XXII. *Circular Measure.*

206. In *practice* angles are always estimated by means of *degrees, minutes* and *seconds;* but there is another method of estimating angles which is very important in *theory*, which we will now explain. The object of the present Chapter is to establish and apply the following proposition: *If with the point of intersection of any two straight lines as centre a circle be described with any radius, then the angle contained by the straight lines may be measured by the ratio which the length of the arc of the circle intercepted between the straight lines bears to the radius.*

207. *The circumferences of circles vary as their radii.*

If a regular polygon of any number of sides be inscribed in a circle, and a regular polygon of the *same* number of sides be inscribed in another circle, the perimeters of the polygons are as the radii of the circles. See Art. 138. This is true however great be the number of the sides; and we may assume that by making the number of sides as large as we please the perimeters of the polygons will not differ sensibly from the perimeters of the corresponding circles.

For a fuller exhibition of this demonstration the student may consult Chapter II. of the larger work on Trigonometry.

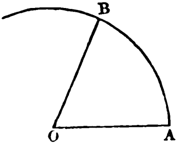
208. Thus the ratio of the circumference of a circle to its radius is *constant* whatever be the magnitude of the circle; therefore of course the ratio of the *circumference* to the *diameter* is also constant. The numerical value of the ratio of the circumference of a circle to its diameter cannot be stated exactly; but it is shewn in the larger work on Trigonometry that the ratio may be calculated to

any degree of approximation. The value is approximately equal to $\frac{22}{7}$, and still more nearly equal to $\frac{355}{113}$; the value correct to eight places of decimals is 3·14159265...

The symbol π is invariably used to denote the ratio of the circumference of a circle to its diameter; hence if r denote the radius of a circle its circumference is $2\pi r$, where

$$\pi = 3{\cdot}14159265...$$

209. *The angle subtended at the centre of a circle by an arc which is equal in length to the radius is an invariable angle.*



With centre O and any radius OA describe a circle; let AB be an arc of this circle equal in length to the radius. Then, since angles at the centre of a circle are proportional to the arcs on which they stand,

$$\frac{\text{angle } AOB}{\text{4 right angles}} = \frac{\text{arc } AB}{\text{circumference of the circle}}$$

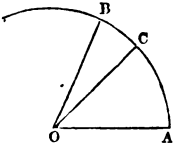
$$= \frac{r}{2\pi r} = \frac{1}{2\pi};$$

therefore angle $AOB = \dfrac{\text{4 right angles}}{2\pi}$.

Thus the angle AOB is a certain fraction of four right angles, which is constant, whatever may be the radius of the circle.

210. Since the angle subtended at the centre of a circle by an arc which is equal to the radius is an *invariable angle* it may be taken as the *unit* of angular measurement, and then any angle will be estimated by the ratio which it bears to this unit.

Let AOC be any angle; with O as centre and any radius OA describe a circle; let AB be an arc of this circle equal in length to the radius; let r denote the radius, and l the length of the arc AC.



Then, since angles at the centre of a circle are proportional to the arcs on which they stand,

$$\frac{\text{angle } AOC}{\text{angle } AOB} = \frac{AC}{AB} = \frac{l}{r};$$

therefore angle $AOC = \frac{l}{r} \times$ angle AOB; this result is true whatever the unit of angular measurement may be, the same unit of course being used for the two angles. If we take the angle AOB itself for the unit, then this angle must be denoted by unity;

$$\text{thus angle } AOC = \frac{l}{r}.$$

211. We have thus shewn that any angle may be estimated by a fraction which has *for its numerator the arc which the angle intercepts on any circle having its centre at the angular point, and for its denominator the radius of that circle.* And in this mode of estimating angles the unit, that is the angle denoted by unity, is the

angle in which the arc intercepted is equal to the radius. We have shewn that this angle is $\frac{4 \text{ right angles}}{2\pi}$; hence the number of *degrees* contained in the angle is $\frac{360}{2\pi}$, that is $\frac{180}{\pi}$. If we use the approximate value of π given in Art. 208, we shall find that $\frac{180}{\pi} = 57 \cdot 29577951\ldots$; this therefore is the number of degrees contained in the angle which is subtended at the centre of a circle by an arc equal to the radius.

212. Thus there are two methods of forming an idea of the magnitude of an angle which is estimated by the fraction *arc divided by radius*. Suppose, for example, we speak of the angle $\frac{3}{4}$: we may refer to the unit of angular measurement which is an angle containing about 57 degrees, and imagine three-fourths of this unit to be taken; or without thinking about the unit at all, we may suppose that an angle is taken such that the arc subtending it is three-fourths of the corresponding radius.

The fraction *arc divided by radius* is called the *circular measure of an angle*. Since, as we have already stated, this method of measuring angles is very much used in theoretical investigations, it is sometimes called the *theoretical* method.

213. If r denote the radius of a circle the circumference is $2\pi r$; hence the circular measure of four right angles is $\frac{2\pi r}{r}$, that is 2π. The circular measure of two right angles is π; the circular measure of one right angle is $\frac{\pi}{2}$; and the circular measure of n right angles is $\frac{n\pi}{2}$, where n may be either integral or fractional.

214. We will now shew how to connect the *circular* measure of any angle with the measure of the same angle in *degrees.*

Let x denote the number of degrees in any angle, θ the circular measure of the same angle. Since there are 180 degrees in two right angles, $\frac{x}{180}$ expresses the ratio of this angle to two right angles. And since π is the circular measure of two right angles, $\frac{\theta}{\pi}$ also expresses the ratio of the angle to two right angles.

Hence $$\frac{x}{180}=\frac{\theta}{\pi};$$

thus $$x=\frac{180\theta}{\pi},$$

and $$\theta=\frac{\pi x}{180}.$$

215. For example, the circular measure of an angle of one degree is $\frac{\pi}{180}$, the circular measure of an angle of three degrees is $\frac{3\pi}{180}$, that is $\frac{\pi}{60}$; the circular measure of an angle of one minute is $\frac{\pi}{180 \times 60}$; the circular measure of an angle of one second is $\frac{\pi}{180 \times 60 \times 60}$; and so on.

Again, if the circular measure of an angle is $\frac{2}{3}$, the number of degrees contained in the angle is $\frac{2}{3} \cdot \frac{180}{\pi}$; that is $\frac{2}{3}$ of $57{\cdot}29577951\ldots$; if the circular measure of an angle is 5, the number of degrees contained in the angle is $5 \,.\, \frac{180}{\pi}$, that is $5 \times 57\cdot29577951\ldots$; and so on.

The student who intends to proceed to the higher parts of mathematics is recommended to pay particular attention to the points illustrated by these examples; especially he should accustom himself to express readily in circular measure an angle which is given in degrees.

216. Similarly we may connect the circular measure of any angle with the measure of the same angle in *grades*.

Let y denote the number of grades in any angle, θ the circular measure of the same angle. Since there are 200 grades in two right angles, $\frac{y}{200}$ expresses the ratio of this angle to two right angles. And since π is the circular measure of two right angles, $\frac{\theta}{\pi}$ also expresses the ratio of the angle to two right angles.

Hence $$\frac{y}{200} = \frac{\theta}{\pi};$$

thus $$y = \frac{200\theta}{\pi},$$

and $$\theta = \frac{\pi y}{200}.$$

The number of grades in the angle which is the unit of circular measure is $\frac{200}{\pi}$, that is $63\cdot661977\ldots$.

EXAMPLES. XXII.

Express the following three angles in circular measure:

1. $\frac{5}{8}$ of a right angle. 2. $24^{0}\,30'$. 3. $50''$.

Find the number of degrees in the following three angles which are expressed in circular measure:

4. $2\frac{1}{2}$. 5. $\cdot 6$. 6. $\cdot 375$.

7. If the length of an arc of a circle which subtends 3^{0} at the centre be 6 feet, find the radius, and the length of an arc subtending 3 grades.

8. Find the number of degrees in the angle subtended at the centre by an arc of 7 feet when the radius is 10 feet.

9. The radius of a circle is 20 feet: determine the length of an arc which subtends an angle of 30^{0} at the centre.

10. The circular measure of the difference of the two acute angles of a right-angled triangle is $\frac{\pi}{18}$: express the two angles in degrees.

11. The angles of a triangle are in Arithmetical Progression; and the circular measure of the Common Difference is $\frac{\pi}{9}$: determine the angles.

12. Find the circular measure of the angle which is the difference of m degrees and n grades.

13. Compare the angle $a^{0}\,b'$ with the angle $a'b^{\backprime}$.

14. If n be any whole number, shew that $\frac{n}{200}$ of a right angle contains a whole number both of English minutes and of French minutes.

15. What must be the unit angle if the sum of the measures of a degree and a grade is 1?

16. An angle is made up of two parts, one containing a degrees and the other containing b grades: compare the angle with a right angle.

17. If $A+B+C=180^0$, shew that

$$\tan A+\tan B+\tan C=\tan A\tan B\tan C.$$

18. If the tangents of the three angles of a triangle be as the numbers 1, 2, 3, shew that they must be equal to 1, 2, 3.

19. Having given $\tan(a+\theta)\tan(a-\theta)=k$, find $\sin\theta$.

20. If $\cos A\cos 3A=\frac{5}{18}$, then $\sin^2 A=\frac{1}{6}$.

21. If $\cos(s-2\alpha)+\cos(s-2\beta)=\cos(s-2\gamma)+\cos(s-2\delta)$,

$$\text{where } s=\alpha+\beta+\gamma+\delta,$$

$$\text{then } \tan\alpha\tan\beta=\tan\gamma\tan\delta.$$

22. Shew that

$$2\cos 5^0\,37'30''=\sqrt{[2+\sqrt{\{2+\sqrt{(2+\sqrt{2})}\}}]}.$$

23. If in a triangle $3\tan\frac{C}{2}=\cot\frac{A}{2}$, shew that the sides are in Arithmetical Progression.

24. The triangle in which

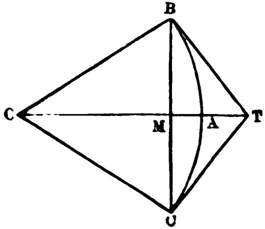
$$\frac{5-4\cos A}{5-4\cos C}=\frac{\sin^2 A}{\sin^2 C}$$

is either isosceles, or such that the sides are in Arithmetical Progression.

XXIII. *Area of a Circle.*

217. The principal object of the present Chapter is to find an expression for the area of a circle; we shall give some important preliminary propositions.

218. *If θ be the circular measure of a positive angle less than a right angle, θ is greater than $\sin\theta$ and less than $\tan\theta$.*



Let AOB be an angle less than a right angle, and let $OB=OA$; from B draw BM perpendicular to OA and produce it to C so that $MC=MB$; draw BT at right angles to OB meeting OA produced at T, and join CT.

Then the triangles MOC and MOB are equal in all respects, so that the angle $TOC=$ the angle TOB; therefore the triangles TOC and TOB are equal in all respects, so that TCO is a right angle, and $TC=TB$.

With centre O and radius OB describe an arc of a circle BAC; this will touch BT at B and CT at C.

Now we assume as an axiom that the straight line BC is less than the arc BAC; thus BM, the half of BC, is less than BA, the half of the arc BAC; therefore $\frac{BM}{OB}$ is less than $\frac{BA}{OB}$; that is, the sine of AOB is less than the circular measure of AOB.

Again, we assume as an axiom that the arc BAC is less than the sum of the two exterior straight lines BT and TC; thus BA is less than BT; therefore $\frac{BA}{OB}$ is less than $\frac{BT}{OB}$; that is, the circular measure of AOB is less than the tangent of AOB.

Hence $\sin\theta$, θ, and $\tan\theta$ are in ascending order of magnitude if θ be less than $\frac{\pi}{2}$.

219. *The limit of* $\frac{\sin\theta}{\theta}$ *when* θ *is indefinitely diminished is unity.*

For $\sin\theta$, θ, and $\tan\theta$ are in ascending order; divide by $\sin\theta$; therefore 1, $\frac{\theta}{\sin\theta}$, and $\frac{1}{\cos\theta}$ are in ascending order of magnitude. Thus $\frac{\theta}{\sin\theta}$ lies in value between 1 and $\frac{1}{\cos\theta}$; but when θ is zero, $\cos\theta$ is unity; hence as θ diminishes indefinitely $\frac{\theta}{\sin\theta}$ approaches the limit unity. Therefore also $\frac{\sin\theta}{\theta}$ approaches the limit unity. And as $\frac{\tan\theta}{\theta} = \frac{\sin\theta}{\theta} \times \frac{1}{\cos\theta}$, the limit of $\frac{\tan\theta}{\theta}$ when θ is indefinitely diminished is also unity.

220. *To find the area of a circle.*

Let r be the radius of a circle. Suppose a regular polygon of n sides described about the circle. Then the circular measure of the angle which each side subtends at the centre of the circle is $\frac{2\pi}{n}$; and therefore, by Art. 139, the area of the polygon is $nr^2 \tan\frac{\pi}{n}$.

$$\text{Now } nr^2 \tan\frac{\pi}{n} = \frac{\pi r^2}{\cos\frac{\pi}{n}} \frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}.$$

Suppose n to increase without limit, then the area of the polygon approximates continually to the area of the circle as the limit, and therefore the area of the circle will be equal to the limit of the above expression.

But when n is indefinitely great

$$\cos\frac{\pi}{n} = 1, \qquad \frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}} = 1; \text{ (Art. 219)}$$

therefore the *area of a circle of radius* $r = \pi r^2$.

221. *To find the area of a sector of a circle.*

Let θ be the circular measure of the angle of the sector; then, by Euclid VI. 33,

$$\frac{\text{area of sector}}{\text{area of circle}} = \frac{\theta}{2\pi};$$

$$\text{therefore area of sector} = \pi r^2 \times \frac{\theta}{2\pi} = \frac{r^2\theta}{2}.$$

Thus the area of a sector is equal to *half the product of the square of the radius into the circular measure of the angle.*

Since θ is the circular measure of the angle the length of the arc of the sector is $r\theta$; hence the area of a sector is equal to *half the product of the length of the arc into the radius.*

EXAMPLES. XXIII.

1. With the angular points of an equilateral triangle as centres, and a radius equal to half the side, arcs are described touching each other: determine the area of the figure which they form.

2. A chord of length r is placed in a circle of radius r: determine the areas of the two segments into which the chord divides the circle.

3. A circle is described round a triangle the angles of which are 45°, 60°, and 75° respectively: determine the areas of the segments of the circle cut off by the sides.

4. Two circles touch each other, and a common tangent is drawn. Supposing their radii to be r and $3r$ respectively, shew that the area of the curvilinear triangle bounded by the two circles and the common tangent is

$$\left(4\sqrt{3}-\frac{11\pi}{6}\right)r^2.$$

5. From the formula $\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$, shew that $\cos\theta$ is greater than $1-\frac{\theta^2}{2}$.

6. If $\frac{\cos\theta}{\theta}+\frac{\theta}{\cos\theta}$ has its least possible arithmetical value, shew that θ is greater than $\sqrt{3}-1$.

7. Shew by Arts. 104 and 188 that in any triangle

$$c=(a-b)\frac{\cos\frac{C}{2}}{\sin\frac{A-B}{2}}=(a+b)\frac{\sin\frac{C}{2}}{\cos\frac{A-B}{2}}.$$

8. The vertical angle of a triangle is 120^0, and the difference of the sides is equal to $\frac{4}{9}$ of the base: find the other angles.

$L\sin 12^0\,50' = 9{\cdot}3465794$, $\log 2 = {\cdot}3010300$.

$L\sin 12^0\,51' = 9{\cdot}3471336$, $\log 3 = {\cdot}4771213$.

9. One of the angles of a plane triangle is 60^0, and the side opposite is to the difference of the two sides including it as 9 is to 2: find the other angles.

$L\cos 78^0\,54'\,10'' = 9{\cdot}2843730$, $\log 3 = {\cdot}4771213$,

$L\cos 78^0\,54'\,20'' = 9{\cdot}2842656$.

10. In a triangle $a = 19$, $b = 1$, $A - B = 90^0$: find C.

$L\tan 41^0\,59' = 9{\cdot}9541834$, $\log 3 = {\cdot}4771213$,

$L\tan 42^0 = 9{\cdot}9544374$.

11. A person standing in the same plane with two vertical poles, and at a distance from the nearer equal to the distance a between them, sees their summits in the same direction. After walking in a straight horizontal line b feet towards the nearer pole he observes that the altitude of one summit is double that of the other. Determine the heights of the two summits.

12. From the deck of a ship which is sailing due North a lighthouse is observed due East, and the altitude of its summit is found to be $12^0\,26'$. After the ship has sailed ten miles the lighthouse is again observed, and its altitude is found to be $7^0\,17'$. Find how far the lighthouse was distant from the ship at the first observation.

$L\sin 5^0\,9' = 8{\cdot}9530996$, $L\sin 7^0\,17' = 9{\cdot}1030373$,

$L\sin 19^0\,43' = 9{\cdot}5281053$, $L\sin 77^0\,34' = 9{\cdot}9896932$,

$\log 7{\cdot}1142 = {\cdot}8521261$, $\log 7{\cdot}1143 = {\cdot}8521322$.

XXIV. *Inverse Notation.*

222. The equation $\sin x = a$ asserts that x is an angle of which the sine is a; it is found convenient to be able to express this relation also in another way, in which x stands alone. The notation used is this, $x = \sin^{-1} a$. Similarly $x = \cos^{-1} a$ expresses that x is an angle of which the cosine is a; and $x = \tan^{-1} a$ expresses that x is an angle of which the tangent is a; and so on.

223. Any relation which has been established among Trigonometrical Ratios may be expressed by means of the inverse notation. Thus, for example, we know that

$$\cos 2\theta = 2\cos^2\theta - 1;$$

this may be written

$$2\theta = \cos^{-1}(2\cos^2\theta - 1):$$

suppose that $\cos\theta = a$, so that $\theta = \cos^{-1} a$,

thus $$2\cos^{-1} a = \cos^{-1}(2a^2 - 1).$$

Again, we know that

$$\sin 2\theta = 2\sin\theta\cos\theta;$$

this may be written

$$2\theta = \sin^{-1}(2\sin\theta\cos\theta):$$

suppose that $\sin\theta = a$, so that $\theta = \sin^{-1} a$, and $\cos\theta = \sqrt{1 - a^2}$,

thus $$2\sin^{-1} a = \sin^{-1}(2a\sqrt{1 - a^2}).$$

224. Also any relation which is expressed in the inverse notation may be converted into a relation expressed in the ordinary notation. Thus, for example, suppose we have given that

$$2\tan^{-1} a = \tan^{-1}\frac{2a}{1 - a^2};$$

take the tangents of both sides; thus

$$\tan(2\tan^{-1} a) = \frac{2a}{1 - a^2}:$$

suppose that $\tan^{-1}a=\theta$, so that $a=\tan\theta$,

thus $$\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}.$$

225. As an example of the inverse notation suppose we require the value of $\sin(\sin^{-1}a+\cos^{-1}b)$.

Let $\sin^{-1}a=A$, and $\cos^{-1}b=B$; then the proposed expression becomes

$$\sin(A+B) \text{ or } \sin A\cos B+\cos A\sin B,$$

now $$\sin A=a,\ \cos A=\surd(1-a^2),$$

$$\cos B=b,\ \sin B=\surd(1-b^2);$$

therefore $$\sin(\sin^{-1}a+\cos^{-1}b)=ab+\surd(1-a^2)\surd(1-b^2).$$

For a numerical illustration take $a=\frac{1}{2}$, and $b=\frac{1}{2}$;

therefore $$\sin\left(\sin^{-1}\frac{1}{2}+\cos^{-1}\frac{1}{2}\right)=\frac{1}{4}+\frac{3}{4}=1.$$

We may express this relation also thus,

$$\sin^{-1}\frac{1}{2}+\cos^{-1}\frac{1}{2}=\sin^{-1}1.$$

Since $\sin\frac{\pi}{2}=1$ we have as *one* possible result,

$$\sin^{-1}\frac{1}{2}+\cos^{-1}\frac{1}{2}=\frac{\pi}{2}.$$

Examples, like the result just given, are often proposed for exercise; but it should be remembered that $\sin^{-1}\frac{1}{2}$ and $\cos^{-1}\frac{1}{2}$ both have an infinite number of values, and thus $\frac{\pi}{2}$ is merely *one* out of an infinite number of possible values of the left-hand member. See Chapter XVIII.

EXAMPLES. XXIV.

1. If $\sin\theta = a$, express in the inverse notation

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta.$$

2. If $\tan\theta = a$, express in the inverse notation

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}.$$

3. Shew that $\cos(2\tan^{-1}a) = \dfrac{1-a^2}{1+a^2}$.

4. Express without inverse notation

$$\sin^{-1}\tan\frac{\pi}{4} \text{ and } \tan\sin^{-1}\frac{1}{\sqrt{2}}.$$

5. Shew that $\tan^{-1}\dfrac{1}{2} + \operatorname{cosec}^{-1}\sqrt{10} = \dfrac{\pi}{4}$.

6. Shew that $2\tan^{-1}\dfrac{2}{3} - \operatorname{cosec}^{-1}\dfrac{5}{3} = \sin^{-1}\dfrac{33}{65}$.

7. In any right-angled triangle, in which C is the right angle,

$$\cot^{-1}\sqrt{\frac{c+a}{c-a}} + \cot^{-1}\sqrt{\frac{c+b}{c-b}} = \frac{\pi}{4}.$$

8. Shew that $2\tan^{-1}\dfrac{1}{2} + \cos^{-1}\dfrac{4}{5} = \dfrac{\pi}{2}$.

9. Shew that

$$\sin^{-1}(\cos x) + \cos^{-1}(\sin y) + x + y = \pi.$$

10. If $\tan^{-1}x + \tan^{-1}3x = \tan^{-1}\dfrac{1}{2}$ find x.

11. If $\tan^{-1}\dfrac{a+x}{a} + \tan^{-1}\dfrac{a-x}{a} = \dfrac{\pi}{6}$, shew that

$$x^2 = 2a^2\sqrt{3}.$$

12. If $\dfrac{\pi}{2} = \sin^{-1}x + \tan^{-1}x$, shew that

$$x^2 = \frac{\sqrt{5}-1}{2}.$$

MISCELLANEOUS EXAMPLES.

1. THE difference of two angles of an isosceles triangle is 20 grades: determine all the angles in degrees.

2. Find the simplest value of x from $\sin 4x = \cos 5x$.

3. Find $\sin x$ from the equation

$$4 \sin x + 3 \cos x = 5.$$

4. If $\sin x + \cos x = \sin A + \cos A$, shew that $\sin x$ must be equal to $\sin A$ or to $\cos A$.

5. Find the greatest value of $\sin x \cos x$.

6. Find the least value of $\tan x + \cot x$.

7. In a triangle $\sin^2 C = \sin^2 A + \sin^2 B$: shew that the triangle is right-angled.

8. One angle of a triangle exceeds the difference of the other two by 60 degrees, and exceeds the smaller of the other two by 50 grades: find the angles.

9. One of the angles of a plane triangle is 120^0, and the sides including it are in the ratio of 4 to 1: shew that the cotangents of the other angles are $3\sqrt{3}$ and $\frac{\sqrt{3}}{2}$.

10. Shew that $\frac{1 + \sin 2A}{1 + \cos 2A} = \frac{1}{2}(1 + \tan A)^2$.

11. Shew that

$$\frac{\sin 3A + \cos 3A}{\sin 3A - \cos 3A} = \frac{2 \sin 2A + 1}{2 \sin 2A - 1} \times \tan(45^0 - A).$$

12. Given

$\log 75{\cdot}6 = 1{\cdot}8785218$, find $\log {\cdot}756$ and $\log({\cdot}0756)^{\frac{1}{3}}$.

13. Find the value of

$$a\cos 2\theta + b\sin 2\theta \quad \text{when } \tan\theta = \frac{b}{a}.$$

14. The sides a, b, c of a triangle are as the numbers 4, 5, 6; find the angle B.

$$\log 2 = \cdot 3010300, \qquad L\cos 27^\circ 53' = 9\cdot 9464040,$$
$$L\cos 27^\circ 54' = 9\cdot 9463371.$$

15. Shew that

$$\sin A\cos(B-C) - \sin B\cos(A-C) = \sin(A-B)\cos C.$$

16. If $\cos(\alpha-\beta)\sin(\gamma-\delta) = \cos(\alpha+\beta)\sin(\gamma+\delta)$, then $\cot\delta = \cot\alpha\cot\beta\cot\gamma$.

17. If $A+B+C=90^\circ$, then

$$\frac{\cos A + \sin C - \sin B}{\cos B + \sin C - \sin A} = \frac{1+\tan\frac{1}{2}A}{1+\tan\frac{1}{2}B}.$$

18. Find the other angles in a triangle when

$$A = 6^\circ 37' 24'' \text{ and } 3c = 7b,$$

$$\log 2 = \cdot 3010300, \qquad L\tan 8^\circ 13' 50'' = 9\cdot 1603083,$$
$$L\tan 3^\circ 18' 42'' = 8\cdot 7624080, \quad L\tan 8^\circ 14' = 9\cdot 1604569.$$

19. If $A+B=90^\circ$, then $\sin(A-B) = -\cos 2A$ and $\sin 2A + \sin 2B = 2\cos(A-B)$.

20. The hypotenuse c of a right-angled triangle ABC is trisected at the points D, E: shew that if CD, CE be joined the sum of the squares on the sides of the triangle $CDE = \frac{2c^2}{3}$.

21. If x, y, z be the lengths of the straight lines bisecting the angles of a triangle, and terminated by the opposite sides a, b, c; shew that

$$(b+c)^2\frac{x^2}{bc} + (c+a)^2\frac{y^2}{ac} + (a+b)^2\frac{z^2}{ab} = (a+b+c)^2.$$

22. The sides AB, BC, CD, DA of a quadrilateral figure inscribed in a circle are in a geometrical progression, of which the common ratio is r; shew that

$$\frac{\tan ABC}{\tan BCD} = \frac{r^2-1}{r^2+1}.$$

23. The tangents of the angles of a certain triangle are 1, 2, 3: if p_1, p_2, p_3 be the perpendiculars on the sides from the opposite angles, shew that $5p_1p_2p_3 = 3abc$.

24. Shew that $\tan(A+30^\circ)\tan(A-30^\circ) = \dfrac{1-2\cos 2A}{1+2\cos 2A}$.

25. If $A+B+C=180^\circ$,

$$\frac{\cot\frac{A}{2}+\cot\frac{C}{2}}{\cot\frac{B}{2}+\cot\frac{C}{2}} = \frac{\sin B}{\sin A}.$$

26. At noon a column in the E.S.E. cast upon the ground a shadow the extremity of which was in the direction N.E.; the angle of elevation of the column being a, and the distance of the extremity of the shadow from the column c, determine the height of the column.

27. Eliminate θ from the equations

$$a\tan\theta + b\sec\theta = c,$$

$$a'\cot\theta + b'\operatorname{cosec}\theta = c'.$$

28. Shew that

$$1+\cos(2A-2B)\cos 2B = \cos^2 A + \cos^2(A-2B).$$

29. Shew that

$$\cos(36^\circ+A)\cos(36^\circ-A)+\cos(54^\circ+A)\cos(54^\circ-A) = \cos 2A.$$

30. In a right-angled triangle CD and CE are drawn from the right angle to the hypotenuse making angles a

and β with the hypotenuse, of which β is the greater: shew that the area of the triangle CDE is

$$\frac{a^2b^2}{2c^2}(\cot\alpha-\cot\beta).$$

31. Solve $8^x.125^{2-x}=2^{4x+3}.5^x$, having given

$$\log 2=\cdot3010300.$$

32. Solve the equation

$$\tan(x+45^\circ)+\tan(x-45^\circ)=2\tan 60^\circ.$$

33. Solve the equation

$$2+\cot^2x=3\sec^4x-\tan^2x.$$

34. If $\sin x=\sin\alpha\sin(x+y)$, then

$$\tan x=\frac{\sin\alpha\sin y}{1-\sin\alpha\cos y}.$$

35. Eliminate θ between the equations

$$a\sin\theta+b\cos\theta=h,$$
$$a\cos\theta-b\sin\theta=k.$$

36. Find $\sin x$ and $\cos x$ if $a\sin x+b\cos x=\sqrt{(a^2+b^2)}$.

37. Given $\log 5=\cdot6989700$, $\log 7=\cdot8450980$, find the logs of 4, 3·5, ·0014, and $\frac{28}{125}$.

38. Given log 2 and log 3, find the logs of 27, 36, 54, ·0025, and $\frac{10}{81}$.

39. The sides of a triangle are respectively 31, 24, and 11 feet: determine the greatest angle.

40. If $\tan x=\cos\alpha\tan y$, then

$$\tan(y-x)=\frac{\tan^2\frac{\alpha}{2}\sin 2y}{1+\tan^2\frac{\alpha}{2}\cos 2y}.$$

41. Shew that
$$\sin^2(A+B)+\cos^2(A-B)=1+\sin 2A\sin 2B.$$

42. Shew that
$$\cos^2 5A=\cos 4A\cos 6A+\sin^2 A.$$

43. Shew that
$$\tan^2\frac{A}{2}=\frac{2\sin A-\sin 2A}{2\sin A+\sin 2A}.$$

44. Shew that $\log_b a$ has always the same sign as $(a-1)(b-1)$.

45. If $\cos A=\sqrt{\frac{2}{3}}$, and $\cos B=\frac{1}{\sqrt{2}}+\frac{1}{2\sqrt{3}}$, shew that $\sin(A-B)=\frac{1}{2}$.

46. Shew that
$$\cos 9A+3\cos 7A+3\cos 5A+\cos 3A=8\cos^3 A\cos 6A.$$

47. A triangular field has its sides respectively 50, 60, 70 yards in length: find its area, having given

$\log 2=\cdot 3010300$, $\log 1\cdot 469=\cdot 1670218$,

$\log 3=\cdot 4771213$, $\log 1\cdot 470=\cdot 1673173$.

48. If A, B, C are the angles of a triangle,
$$\cos\frac{A}{2}+\cos\frac{B}{2}\text{ is greater than }\cos\frac{C}{2}.$$

49. If A, B, C are the angles of a triangle,
$$\sin\frac{A}{2}+\sin\frac{B}{2}\text{ is greater than }\cos\frac{C}{2}.$$

50. If A, B, C are the angles of an acute-angled triangle, $\sin\frac{A}{2}+\sin\frac{B}{2}$ is greater than $\sin\frac{C}{2}$.

51. Express according to the French measure $22^\circ 42' 9''$ and $65^\circ 27' 9''$.

52. Given $\log 2 = \cdot 3010300$ and $\log 3 = \cdot 4771213$, find $\log 324$ and $\log \cdot 015$.

53. Two straight lines which cross one another meet a canal at angles of 30° and 60° respectively. If it be three miles by the longer of the two roads from the crossing to the canal, find the distance by the shorter. If there be a foot-path which goes the shortest way to the canal, find the distance by it.

54. Given $a = 200$, $B = 45^\circ$, $C = 10^\circ$: find b.

$$\log 2 = \cdot 3010300, \quad \log 172\cdot 64 = 2\cdot 2371414,$$

$$L \sin 55^\circ = 9\cdot 9133645, \quad \log 172\cdot 65 = 2\cdot 2371666.$$

55. From the top of a hill I observe two cottages in a straight horizontal line before me. I find their angles of depression to be 45° and 30° respectively, and know them to be 176 yards apart. Find the height of the hill.

56. Shew that in any triangle

$$\frac{a - b\cos C}{ab^2} + \frac{b - c\cos A}{bc^2} + \frac{c - a\cos B}{ca^2} = \frac{a^4 + b^4 + c^4}{2a^2b^2c^2},$$

$$(a\sin A + b\sin B + c\sin C)^2 = (a^2 + b^2 + c^2)(\sin^2 A + \sin^2 B + \sin^2 C).$$

57. Shew that if the sides a, b, c of a triangle are in Arithmetical Progression

$$\frac{\sin(C - B)}{\sin(C + A)} = \frac{(3c + a)(c - a)}{2a(c + a)}.$$

58. A water-wheel whose diameter is 12 feet makes 30 revolutions per minute: find approximately the number of miles per hour traversed by a point in the circumference of the wheel.

59. Solve the equation

$$\sin 3x = 2\sin 2x - \sin x.$$

60. Shew that

$$\tan^{-1}\frac{\sqrt{3}+1}{\sqrt{3}-1}-\tan^{-1}\frac{1}{\sqrt{3}}=45^\circ.$$

61. If $\tan 2A=\frac{2n\sqrt{(1-n^2)}}{1-2n^2}$, find $\sin A$.

62. Given $b=65$, $c=35$, $A=85^\circ$, find B and C.

$$\log 3 = \cdot 4771213, \quad L\tan 18^\circ 7' = 9\cdot 5147766,$$
$$L\cot 42^\circ 30' = 10\cdot 0379475, \quad L\tan 18^\circ 8' = 9\cdot 5152039.$$

63. Given $c=\sqrt{60}$, $b=\sqrt{120}$, $B=135^\circ$: find A and a.

64. If in a triangle

$$\sin A : \sin C :: \sin(A-B) : \sin(B-C),$$

then the squares of the sides are in Arithmetical Progression.

65. Shew with the notation of Chapter XIV. that

$$\frac{r_1-r_2}{ar_1-br_2}=\frac{r_2-r_3}{br_2-cr_3}=\frac{r_3-r_1}{cr_3-ar_1}=\frac{1}{s}.$$

66. Shew that

$$\frac{\sin A+\sin 5A}{\sin 2A+\sin 4A}=\cos 2A\sec A,$$

$$\sin 3A+\cos 3A=(\cos A-\sin A)(1+4\sin A\cos A).$$

67. If $2\sec\theta=\sec(\theta+2\alpha)+\sec(\theta-2\alpha)$, shew that

$$\cos^2\theta=2\cos^2\alpha.$$

68. Solve the equation $\sin x+\cos x=1$.

69. Taking the diameter of the earth as 8000 miles, find the angle subtended at the Earth's centre by an arc of the Equator 500 miles long.

70. Shew that

$$\frac{1}{2}\sin^{-1}\frac{4}{5}+\tan^{-1}\frac{1}{3}=45^\circ.$$

71. Express in degrees, in grades, and in circular measure the difference between a degree and a grade.

72. The sides of a triangle are $m+n$, $m-n$, and $\sqrt{2(m^2+n^2)}$, and the sine of one angle is $\frac{\sqrt{5}-1}{4}$: find the other angles.

73. Given $a=15$, $b=35$, $B=120^\circ$: find A.

$\log 3 = \cdot 4771213$, $L \sin 21^\circ 47' = 9\cdot 5694883$,
$\log 14 = 1\cdot 1461280$, $L \sin 21^\circ 48' = 9\cdot 5698043$.

74. Given $a=65$, $b=16$, $C=60^\circ$: find A and B.

$\log 3 = \cdot 4771213$, $L \tan 46^\circ 20' = 10\cdot 0202203$,
$\log 7 = \cdot 8450980$, $L \tan 46^\circ 21' = 10\cdot 0204732$.

75. A person walking along a straight road watches two spires until they appear in the same straight line, and finds that this straight line makes an angle β with the road. From the spot where this is the case he walks a distance c in yards, when the nearest spire lies in a direction at right angles to that of the road, and he observes that at this point the angle subtended by the two spires is α. Shew that the distance between the spires in yards is

$$\frac{c\cos\alpha}{\cos(\alpha+\beta)}-\frac{c}{\cos\beta}.$$

76. The sides of a triangle are in Arithmetical Progression and the area is four-fifths of the area of an equilateral triangle having the same perimeter: shew that the sides are as the numbers 7, 10, 13.

77. Shew that

$$\frac{\tan(45^\circ+A)}{\tan(45^\circ-A)}=\frac{2\cos A+\sin A+\sin 3A}{2\cos A-\sin A-\sin 3A}.$$

78. Solve the equation

$$\sin^2 x + \sin x = \cos^2 x + \cos x.$$

79. In a plane triangle the lengths of two sides are 169 and 125 yards respectively, and the included angle $= 2\sin^{-1}\frac{33}{65}$: find the third side and the area.

80. Shew that $\sin\left(\tan^{-1}\frac{3}{4} + \cot^{-1}\frac{5}{12}\right) = \frac{63}{65}$.

81. The angles of a triangle are in the proportion of 1, 2, and 3: express the angles in grades and degrees.

82. Solve the equation

$$\sqrt{3}\tan^2\theta - (1 + \sqrt{3})\tan\theta + 1 = 0.$$

83. Given $a = 8$, $b = 7$, $A = 120^0$: find B and C.

$\log 7 = {\cdot}8450980$, $\quad L\sin 60^0 = 9{\cdot}9375306$,

$\log 8 = {\cdot}9030900$, $\quad L\sin 49^0\,16' = 9{\cdot}8795287$.

$L\sin 49^0\,17' = 9{\cdot}8796375$,

84. Given $b = 50$, $c = 30$, $A = 30^0$: find B and C.

$\log 2 = {\cdot}3010300$, $\quad L\tan 43^0 = 9{\cdot}9696559$,

$L\cot 15^0 = 10{\cdot}5719475$, $\quad L\tan 43^0\,1' = 9{\cdot}9699091$. .

85. Given $\log 2 = {\cdot}3010300$, and $\log 3 = {\cdot}4771213$, find $\log\sqrt{5}$, $\log\sqrt[3]{{\cdot}0020736}$, $\log 187{\cdot}5$, and $\log {\cdot}00288$.

86. Shew that

$$\cos^2 18^0 \cos 36^0 - \sin 18^0 \sin^2 36^0 = \frac{5}{8}.$$

87. Shew that

$$\frac{\cos 2A - \cos 5A}{\cos 3A - \cos 4A} = 3\cos^2\frac{A}{2} - \sin^2\frac{A}{2}.$$

88. Shew that

$$\tan(A+B)+\tan(A-B) = \tan 2A\,\{1-\tan(A-B)\tan(A+B)\}.$$

89. Solve the equation $\sin 2x = 3\tan x\cos 2x$.

90. Shew that

$$\sin 5A - 5\sin 3A + 10\sin A = 16\sin^5 A.$$

91. If $\dfrac{\sin\theta}{x} = \dfrac{\cos\theta}{y}$, and $\dfrac{\cos^2\theta}{x^2} + \dfrac{\sin^2\theta}{y^2} = \dfrac{10}{3(x^2+y^2)}$, find $\tan\theta$.

92. Find A, B, and C from the equations

$$\cos(A+B-C) = \frac{1}{2},\ \cos(A-B+C) = \frac{\surd 3}{2},$$
$$\cos(A+B) = \sin C.$$

93. Shew that

$$\cos 7A + \cos 5A + \cos 3A + \cos A = 4\cos 4A\cos 2A\cos A.$$

94. Find all the values of θ less than 180° for which $\sin 3\theta = \sin 30^\circ$.

95. Shew that

$$\sin(A-B)\cos(A+B) + \sin(B-C)\cos(B+C) + \sin(C-A)\cos(C+A) = 0.$$

96. Solve the equation $\tan x = \cos x$.

97. Shew that

$$\operatorname{cosec} 2A + \cot 4A + \operatorname{cosec} 4A = \cot A.$$

98. Given $b=17$, $c=7$, $A=60^\circ$: find B and C.

$\log 2 = \cdot 3010300$, $L\tan 35^\circ 49' = 9\cdot 8583357$,

$\log 3 = \cdot 4771213$, $L\tan 35^\circ 49' 10'' = 9\cdot 8583800$.

99. A staff $2h$ feet high placed on the top of a tower subtends an angle α at a place whose horizontal distance from the foot of the tower is c feet: determine the height of the tower.

100. The sides a, b, c of a triangle are in Arithmetical Progression: shew that the area

$$=\frac{b}{4}\sqrt{3(3b-2a)(2a-b)}.$$

ANSWERS.

I. 1. 60^g. 2. $1^g\cdot5$. 3. 7^g. 4. $10^g\cdot925$.
5. $30^g\cdot775$. 6. $74^g\cdot925$. 7. 27°. 8. $3^\circ 9'$.
9. $9^\circ 22' 57''$. 10. $18^\circ 41' 51''$. 11. $27^\circ 58' 3''$.
12. $68^\circ\cdot8068$. 13. $18^\circ, 20^g$. 14. $56^\circ 15', 62^g\cdot5$.
15. $61^\circ\cdot875, 68^g\cdot75$. 16. $63^\circ\cdot28125, 70^g\cdot3125$. 17. $60^\circ, 66^g\tfrac{2}{3}$.
18. $36^\circ, 40^g$. 19. $18^\circ, 9^\circ$. 20. $54^\circ, 36^\circ$. 21. 54.
22. 32·4. 23. $111\tfrac{1}{9}$. 24. $185\tfrac{5}{27}$. 25. 34 to 27.

II. 1. $\cos A = \frac{5}{13}$. 2. $\cos A = \frac{9}{41}$. 3. $\sin A = \frac{24}{25}$.
4. $\sin A = \frac{11}{61}$. 5. $\sin A = \frac{4}{5}$. 6. $\cos A = \frac{2\sqrt{2}}{3}$.
7. $\cos A = \frac{m^2-1}{m^2+1}$. 8. $\sin A = \frac{m^2-n^2}{m^2+n^2}$.

III. 1. 30°. 2. 60°. 3. 60°. 4. 60°.
5. 45° or 60°. 6. 15° or 75°. 7. $A=45^\circ$, $B=30^\circ$.
8. $A=60^\circ$, $B=45^\circ$. 9. $A=45^\circ$, $B=15^\circ$.
10. $A=18^\circ$, $B=24^\circ$. 11. $A=10^\circ$, $B=5^\circ$.
12. $A=45^\circ$, $B=45^\circ$. 13. $A=15^\circ$, $B=30^\circ$, $C=15^\circ$.

14. $\sin 22\tfrac{1}{2}^\circ = \sqrt{\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)}$, $\cos 22\tfrac{1}{2}^\circ = \sqrt{\left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right)}$,
$\tan 22\tfrac{1}{2}^\circ = \sqrt{\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)}$; it may be reduced to $\sqrt{2}-1$.

15. $\frac{\sqrt{3}-1}{2\sqrt{2}+\sqrt{3}+1}$; it may be reduced to $\sqrt{6}-\sqrt{3}+\sqrt{2}-2$.

16. $\frac{\sqrt{3}+1}{2\sqrt{2}+\sqrt{3}-1}$; it may be reduced to $\sqrt{6}+\sqrt{3}-\sqrt{2}-2$.

IV. 1. $\sin A = \frac{21}{29}$, $\cos A = \frac{20}{29}$.

2. $\cos A = \frac{m(2n+m)}{2n^2+2nm+m^2}$. 3. $\sin A = \frac{pm+qn}{\sqrt{(p^2+q^2)}\sqrt{(m^2+n^2)}}$,
$\cos A = \frac{pn-qm}{\sqrt{(p^2+q^2)}\ \sqrt{(m^2+n^2)}}$. 4. $A = 15^\circ$ or 75°.
5. $A = 30^\circ$ or 60°, $B = 60^\circ$ or 30°.
6. $A = 15^\circ$ or 75°, $B = 75^\circ$ or 15°. 7. $A = 15^\circ$ or 75°,
$B = 75^\circ$ or 15°. 9. $\frac{100}{\sqrt{3}}$ feet. 10. $50(3-\sqrt{3})$ yards.
11. $10(3+2\sqrt{3})$ feet, $10\sqrt{3}$ feet. 12. $\frac{a^2\sqrt{3}}{4}$.
13. $\frac{a^2}{4}$. 14. 27 feet. 15. 1 or 2: assume $2h$ for
the height of the post, then we shall find that the distance of the point of observation is either h or $2h$.
16. $(a+b)(2-\sqrt{3})$. 17. 100 feet.
18. $\frac{h}{h+70} = \frac{1}{3}$ gives $h = 35$; $35 + 5 = 40$.

V. 1. 8. 2. $\frac{5}{2}$. 3. $\frac{5}{2}$. 4. $\frac{4}{3}$. 5. $\frac{3}{2}$. 6. $\frac{7}{5}$.
7. 1·2552726. 8. 1·7781513. 9. 2·3344539.
10. 3·8115752. 11. 3·7323939. 12. $\bar{1}$·6478174.
13. ·6354839. 14. $\bar{1}$·8573326. 15. $\bar{1}$·5740313.
16. $\bar{2}$·4771213. 17. $\bar{1}$·7406162. 18. $\bar{1}$·6365006.
19. 1·7993406. 20. $\bar{1}$·2640466. 21. $\bar{3}$·3555614.
22. ·0511525. 23. ·0880456. 24. ·0263938.
25. ·3521825. 26. 0, 2, $\bar{2}$, $\bar{5}$; $\bar{3}$·1518804.
27. $\bar{1}$·7041854. 28. $-\frac{4}{3}$. 29. $\bar{4}$·4983106. Observe
$315 = \frac{14 \times 15 \times 15}{10}$. 30. $\frac{2-\log 2}{2-2\log 2} = 1{\cdot}2153383$.

31. $\dfrac{3}{2-2\log 2}=2{\cdot}1460148.$

32. $\dfrac{3}{3\log 3+2\log 2-2}=89{\cdot}75613.$

VI. 1. 2·5389050. 2. 6·6180633. 3. 2·8663518.
4. 1·2435557. 5. 2·7855128. 6. 6·7915179.
7. 616·0148. 8. ·007501467. 9. 2.
10. 2·6533. 11. 9·7932666. 12. 9·7299388.
13. 9·8241849. 14. 9·9793629. 15. 9·5906364.
16. 9·7154581. 17. 10·1650011. 18. 10·2618877.
19. 10·0171747, 9·4577109. 20. $16^\circ 19' 26''$.
21. $6^\circ 53' 8''$. 22. $22^\circ 28' 16''$.
23. $80^\circ 52' 51''$. 24. 142·7035.

VII. 1. $a=75$, $b=75\sqrt{3}$. 2. $A=30^\circ$, $b=100\sqrt{3}$.
3. $b=80(2-\sqrt{3})$, $c=80(\sqrt{6}-\sqrt{2})$.
4. $A=45^\circ$, $c=75\sqrt{2}$. 5. $a=97{\cdot}082040$, $b=70{\cdot}534236$.
6. $A=43^\circ 36' 10''$, $b=210$.
7. $b=125(\sqrt{2}-1)$, $c=125\sqrt{(4-2\sqrt{2})}$.
8. $c=5$, $B=53^\circ 7' 48''$. 9. $a=78{\cdot}1548$, $b=179{\cdot}744$.
10. $A=35^\circ 49' 44''$, $b=132{\cdot}966$.
11. $b=59{\cdot}8766$, $c=138{\cdot}24$. 12. $A=36^\circ 9' 3''$, $c=239{\cdot}02$.

VIII. 1. $b=84\sqrt{2}$, $a=42(\sqrt{2}+\sqrt{6})$.
2. $B=30^\circ$, $c=48\sqrt{3}$.
3. $B=45^\circ$ or 135°, $c=\frac{1}{2}(\sqrt{6}+3\sqrt{2})$ or $\frac{1}{2}(\sqrt{6}+\sqrt{2})$.
4. $B=90^\circ$, $c=2\sqrt{2}-\sqrt{6}$. 5. $B=30^\circ$. 6. $A=90^\circ$, $B=60^\circ$.
7. $a=107{\cdot}087$, $b=96{\cdot}5836$. 8. $c=326{\cdot}576$.
9. $B=55^\circ 13' 28''$ or $124^\circ 46' 32''$, $c=265{\cdot}342$ or $87{\cdot}389$.
10. $B=55^\circ 9' 8''$, $c=537{\cdot}079$.
11. $B=40^\circ 56' 39''$, $c=106{\cdot}735$.
12. $A=38^\circ 25' 20''$, $B=57^\circ 41' 24''$.

IX. 1. 108, 120. 2. 75 grades. 3. 30°.

4. $\tan\frac{1}{2}A=\frac{1}{5}$. 6. $\cos^2 A=\frac{1}{4}(2-\sqrt{2})$, $\cos 2A=-\frac{1}{\sqrt{2}}$.

8. $\cos 2A=-\frac{4}{5}$, $\cos A=\frac{1}{\sqrt{10}}$.

10. $\sin^2 A=\frac{m^2(1-n^2)}{m^2-n^2}$, $\sin^2 B=\frac{1-n^2}{m^2-n^2}$.

11. $\sin^2 A=\frac{m^2-n^2}{1-n^2}$, $\sin^2 B=\frac{m^2-n^2}{m^2(1-n^2)}$; also we may have $\sin A=0$, $\sin B=0$. 14. $\sin A=\frac{40}{41}$.

16. $\sin^2 x=\frac{mn-mb-na}{(m-n)(a-b)}$, $\sin^2 y=\frac{m-a-b}{m-n}$.

21. $B=75^\circ\,57'\,50''$. 22. $A=38^\circ\,15'\,8''$.

24. $\pm\left\{a-\sqrt{\left(\frac{1}{2}-a^2\right)}\right\}$.

X. 1. 30° or 150°. 3. $B=45^\circ$. 4. $B=90^\circ$, $c=a\sqrt{3}$.
5. 120°. 6. 90°. 7. 120°. 8. 14, 12. 10. 4·58257.
14. $\pm\{pq-\sqrt{(1-p^2)(1-q^2)}\}$. 17. 30° or 150°.
19. $\frac{m\sin\alpha}{\sin(\beta-\alpha)}$. 20. Substitute in the third equation the values of $\tan^2 x$ and $\tan^2 y$ from the first two; see Example IX. 16: then reduce the result.

XI. 1. 71·0919. 2. 226·623. 3. 1239·632.
4. $B=149^\circ\,20'\,31''$, $C=24^\circ\,39'\,29''$.
5. $B=116^\circ\,33'\,54''$, $C=26^\circ\,33'\,54''$.
6. $B=76^\circ\,44'\,55''$, $C=53^\circ\,59'\,5''$.
7. $B=101^\circ\,29'\,9''$, $C=36^\circ\,0'\,51''$.
8. $B=86^\circ\,34'\,27''$, $C=50^\circ\,15'\,33''$.
9. $B=68^\circ\,2'\,24''$, $C=54^\circ\,4'\,19''\cdot 2$.
10. $B=81^\circ\,2'\,16''$, $C=50^\circ\,37'\,44''$, $a=4\cdot 3485$.

11. $B=23^\circ\, 8'\, 33''$, $C=32^\circ\, 17'\, 27''$.
12. $B=35^\circ\, 15'\, 52''$, $C=84^\circ\, 44'\, 8''$, $c=137{\cdot}9796$.
13. $B=17^\circ\, 6'\, 45''$, $C=133^\circ\, 2'\, 15''$.
14. $B=51^\circ\, 41'\, 20''$, $C=58^\circ\, 6'\, 30''$, $\log c={\cdot}9271876$.
15. $B=41^\circ\, 19'\, 20''$ or $138^\circ\, 40'\, 40''$,
$C=105^\circ\, 58'\, 40''$ or $8^\circ\, 37'\, 20''$.
16. $B=30^\circ$ or 150°.
17. $A=32^\circ\, 57'\, 8''$.
18. $43^\circ\, 2'\, 56''$, $53^\circ\, 46'\, 44''$, $83^\circ\, 10'\, 20''$.
19. $56^\circ\, 15'\, 4''$, $59^\circ\, 51'\, 10''$, $63^\circ\, 53'\, 46''$.
20. $70^\circ\, 53'\, 36''$, 60°, $49^\circ\, 6'\, 24''$.

XII. 1. 72 feet. 2. $\dfrac{\sqrt{3}+1}{\sqrt{2}}$ miles.
3. $500\,(\sqrt{3}+1)$ yards. 4. $18\sqrt{2}$ miles.
5. 93·489 yards. 6. 155·823 feet.
7. 228·6307 yards. 8. 280·015, 765·015 feet.
9. 212·858, 618·186 feet. 10. 238·502, 480·504 yards.
11. E $7^\circ\, 21'\, 45''$ S. 12. $4\sin 67\tfrac{1}{2}^\circ - 4\sin 22\tfrac{1}{2}^\circ$.
13. Distance $=\dfrac{5\sin 45^\circ}{\sin 22\frac{1}{2}^\circ}$; the first ship bears E.N.E of the second. 14. $40\sqrt{330}$ feet per minute.
15. Height 80 feet; distance $4800\sqrt{330}$ feet.
16. If h be the height $3\,(200)^2=h^2\,(4-\sqrt{6})$.
18. Solve CEF knowing EF and the angles; then solve AEC and BFC. 21. 162·95. 22. 1514·396, 4163·746.
23. 509·7. 24. 2109·8.

XIII. 1. 72°. 2. 36°, 54°. 8. $\alpha=45^\circ$, $\beta=30^\circ$.
12. $10\,(\sqrt{2}-1)$, $10\sqrt{4-2\sqrt{2}}$.
13. $AC=\dfrac{h\cos\beta\cos\alpha}{\sin(\alpha-\beta)}$, $CD=\dfrac{h\cos\beta\sin\alpha}{\sin(\alpha-\beta)}$.
14. $\dfrac{C}{2}=41^\circ 24' 35''$, $B=\dfrac{C}{2}$: see Example x. 9.

15. $\frac{C}{2}=57^\circ\,18'\,41''$, $B=\frac{C}{2}$. 16. Let AC and BD meet at O; then the triangles AOB and COD are similar.

18. $\sin^2 A=\frac{c(b-c)}{a(a+b-2c)}$, $\sin^2 B=\frac{c(a-c)}{b(a+b-2c)}$.

XIV. 3. $\frac{a}{2\sqrt{3}}$. 4. $\frac{a}{\sqrt{3}}$. 5. $S=2310$, $r=21$, $R=42\frac{1}{2}$.
6. $6\times(81)^2$. 7. 2456840. 8. 382094.
16. The radius of the circle passing through the three points is $\frac{41\times 13}{8}$; the tower is at the centre of this circle.

XV. 5. $60^\circ, 120^\circ, 240^\circ, 300^\circ$. 6. $22\frac{1}{2}^\circ, 112\frac{1}{2}^\circ, 202\frac{1}{2}^\circ, 292\frac{1}{2}^\circ$.
7. $90^\circ, 210^\circ, 330^\circ$. 8. $120^\circ, 180^\circ, 240^\circ$.
9. $15^\circ, 75^\circ, 195^\circ, 255^\circ$. 10. $67\frac{1}{2}^\circ, 157\frac{1}{2}^\circ, 247\frac{1}{2}^\circ, 337\frac{1}{2}^\circ$.
11. $0^\circ, 30^\circ, 150^\circ, 180^\circ$. 12. $60^\circ, 90^\circ, 270^\circ, 300^\circ$.

XVI. 7. $\tan^2 A+\cot^2 A=(\tan A-\cot A)^2+2$. 8. 4.
11. $12150+6250\sqrt{2}$.

XVII. 1. $-\frac{1}{\sqrt{2}}$. 2. 1. 3. $\frac{\sqrt{3}}{2}$. 4. $-\frac{\sqrt{3}}{2}$.
5. -1. 6. $\frac{1}{2}$. 7. $2-\sqrt{3}$. 8. $-(2-\sqrt{3})$.
9. $\sqrt{3}$. 10. $\frac{1}{\sqrt{3}}$. 11. $-\sqrt{3}$. 12. 1.

XVIII. 7. $A=(4n+1)\,90^\circ$. 8. $90^\circ-2A=n360^\circ\pm 3A$.

XIX. 1. -1. 2. $\frac{\sqrt{3}+\sqrt{15}}{8}$.
4. $\frac{(1-ab)^2-(a+b)^2}{(1+a^2)(1+b^2)}$, $\frac{2(a+b)(1-ab)}{(1+a^2)(1+b^2)}$. 6. $-\frac{1}{2}$.
7. $\frac{3}{5}$. 9. $\frac{p^2(2-c^2)-q^2}{q^2-p^2c^2}$ or $\frac{p^2c^2+q^2(1-2c^2)}{q^2-p^2c^2}$. 11. 74°.

XX. 27 may be deduced from 26; 28 may be deduced from 26 by changing angles into their complements; 29 may by deduced from 28; 30 may be deduced from 26 by changing C into $90^\circ - C$.

36. $x = n180^\circ$, or $x = n180^\circ + (-1)^n 30^\circ$.

37. $4x = x + n180^\circ$.

38. $2x = (2n+1)\,90^\circ$, or $7x = n180^\circ + (-1)^n 30^\circ$.

39. $8x = n180^\circ$, or $8x = n360^\circ \pm 60^\circ$.

40. $(a+b+2c)\,x = n360^\circ \pm (a+b)\,x$. 41. $\sqrt{3}$.

45. $1 \pm \sqrt{\dfrac{2b}{1+b}}$.

XXI. 1. $\dfrac{\sqrt{\{2+\sqrt{(2+\sqrt{2})}\}}}{2}$. 3. $\dfrac{13+\sqrt{31}}{20}$.

XXII. 7. $\dfrac{360}{\pi}$ feet, $\dfrac{54}{10}$ feet. 8. $\dfrac{126}{\pi}$.

9. $\dfrac{10\pi}{3}$ feet. 10. 50°, 40°. 11. 40°, 60°, 80°.

12. $\left(m - \dfrac{9}{10}n\right)\dfrac{\pi}{180}$. 13. $\dfrac{50}{27}\,\dfrac{60a+b}{100a+b}$. 15. $1\frac{9}{10}{}^\circ$.

16. $\dfrac{10a+9b}{900}$. 19. $\sin^2\theta = \dfrac{\sin^2 a - k\cos^2 a}{1-k}$.

XXIII. 1. $\dfrac{a^2}{4}\left(\sqrt{3} - \dfrac{\pi}{2}\right)$, where a is a side of the triangle. 2. The smaller segment is $\dfrac{\pi r^2}{6} - \dfrac{r^2\sqrt{3}}{4}$.

6. We must have $\theta = \cos\theta$, and therefore $1 - \dfrac{\theta^2}{2}$ less than θ. 8. $42^\circ\,50'\,22''$, $17^\circ\,9'\,38''$.

9. $71^\circ\,5'\,45''$, $48^\circ\,54'\,15''$. 10. $83^\circ\,58'\,28''$.

11. Let x be the height of the lower; then that of the other is $2x$; and $4x^2 = (3b-2a)(2a-b)$.

12. 7·11423 miles.

XXIV. 4. $n180^\circ + (-1)^n 90^\circ$, $\tan\{n180^\circ + (-1)^n 45^\circ\}$.

MISCELLANEOUS EXAMPLES.

1. $54^0, 54^0, 72^0$ or $66^0, 66^0, 48^0$. 2. 10^0.

3. $\sin x = \frac{4}{5}$, $\cos x = \frac{3}{5}$. 5. $\frac{1}{2}$. 6. 2. 8. $37\frac{1}{2}^0, 60^0, 82\frac{1}{2}^0$.

13. a. 14. $55^0\,46'\,16''$. 18. $B = 4^0\,55'\,10''\cdot6$, $C = 168^0\,27'\,25''\cdot4$.

26. $\sqrt{2-\sqrt{2}}\,c\tan a$. 27. $(cc'-aa')^2 = (ab'+c'b)^2 + (a'b+cb')^2$.

31. 1·06. 32. $x = (3n+1)30^0$. 33. $x = n180^0 \pm 30^0$.

35. $a^2+b^2 = h^2+k^2$. 39. 120^0. 47. 1469·3.

51. $25^g{\cdot}225, 72^g{\cdot}725$. 52. $2{\cdot}5105452, \bar{2}{\cdot}1760913$. 53. $\sqrt{3}, 1{\cdot}5$.

54. 172·6436. 55. $88(\sqrt{3}+1)$. 58. 12·85...

59. $x = n90^0$. 61. $\pm n$. 62. $65^0\,37'\,41''$, $29^0\,22'\,19''$.

63. $A = 15^0$, $a = \sqrt{30}(\sqrt{3}-1)$. 68. $x - 45^0 = n360^0 \pm 45^0$.

69. $7^0{\cdot}16$... 71. $\frac{1^0}{10}, \frac{1^g}{9}, \frac{\pi}{1800}$. 72. $90^0, 72^0$.

73. $21^0\,47'\,12''$. 74. $A = 106^0\,20'\,12''$, $B = 13^0\,39'\,48''$.

78. $x = n180^0 + 45^0$, or $x - 45^0 = n360^0 \pm 135^0$.

79. 154, 9240. 81. $30^0, 60^0, 90^0$; $33^g\frac{1}{3}, 36^g\frac{2}{3}, 100^g$.

82. $\theta = n180^0 + 45^0$ or $n180^0 + 30^0$. 83. $B = 49^0\,16'\,5''$, $C = 10^0\,43'\,55''$. 84. $B = 118^0\,0'\,55''$, $C = 31^0\,59'\,5''$.

85. $\cdot3494850, \bar{1}{\cdot}1055751, 2{\cdot}2730013, \bar{3}{\cdot}4593926$.

89. $\sin x = 0$ or $\cos 2x = \frac{1}{2}$. 91. $\tan^2\theta = 3$ or $\frac{1}{3}$.

92. $A = 45^0$, $B = 30^0$, $C = 15^0$. 94. $10^0, 50^0, 130^0, 170^0$.

96. $\sin x = \frac{\sqrt{5}-1}{2}$. 98. $B = 95^0\,49'\,3''$, $C = 24^0\,10'\,57''$.

99. $\sqrt{(2ch\cot a - c^2 + h^2)} - h$.

THE END.

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